Probabilities on a Heyting Algebra (ver 2)

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Abstract

Probability is initially defined on a Boolean algebra or a lattice of projectors. We propose here to start from a Heyting algebra.

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1 Introduction

Probabilities are usually defined over Boolean algebras. In the finite case, it is sufficient to give the values for the atoms. We show here that it is possible to generalize them to Heyting algebras, starting from the prime filters, in other words on the spectrum of the algebra. Surprisingly, it is also possible to define them in a quantum manner, using projectors.

2 Affine measures

We place ourselves in a finite Heyting algebra $\mathfrak{H}$. The set of its filters will be denoted $\mathcal{F}$ and the subset of its prime filters Φ, which is the co-spectrum of $\mathfrak{H}$ [1]. The finiteness of the algebra implies that all filters are principal. G will be the subset of prime filter infs. A filter can be seen as a set of propositions validated at a certain time, or as the intension of a formal object.

Definition

An *affine measure* on $\mathfrak{H}$ is an application $\mu : \mathfrak{H} \to \mathbb{R}$ that verifies

$$(i) \quad \mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$$

[1]The term spectrum is usually reserved for ideals.

If, in addition, we have $\mu(\perp) = 0$, $\mu(\top) = 1$ and $a \leqslant b \Longrightarrow \mu(a) \leqslant \mu(b)$, it becomes a *probability*.

Let $t = (a, b)$. It is useful to introduce a deviation matrix translating relation (i) with Kronecker indices:

$$\Delta_c^t = \delta_c^{a \vee b} + \delta_c^{a \wedge b} - \delta_c^a - \delta_c^b$$

We can see that the affine measures verify $\Delta\mu = 0$ [2], i.e. $\mu \in Ker\,\Delta$.

Let $O_a^b = (a \leqslant b) = \begin{cases} 1 & if\ a \leqslant b \\ 0 & otherwise \end{cases}$ be the matrix of the order relation of $\mathfrak{H}$.

Proposition

$$\forall t. \Delta_a^t O_f^a = 0 \Longleftrightarrow f \in G\ or\ f = \perp\ or\ f = \top$$

Demonstration

The relation on the left is $O_f^{a \wedge b} + O_f^{a \vee b} - O_f^a - O_f^b = 0$, or $(a \wedge b \geqslant f) + (a \vee b \geqslant f) = (a \geqslant f) + (b \geqslant f)$.

If $\uparrow f$ is prime, we have $f \leqslant a \vee b \Longleftrightarrow f \leqslant a\ or\ f \leqslant b$, and $f \leqslant a \wedge b \Longleftrightarrow f \leqslant a\ and\ f \leqslant b$.

Apart from the trivial cases $f = \perp\ or\ \top$, let's look at the various possibilities:

$f \leqslant a$	$f \leqslant b$	$(f \leqslant a) + (f \leqslant b)$	$(f \leqslant a \vee b) + (f \leqslant a \wedge b)$
0	0	$0 + 0$	$0 + 0$
0	1	$0 + 1$	$1 + 0$
1	0	$1 + 0$	$1 + 0$
1	1	$1 + 1$	$1 + 1$

And if $f \notin G$, we find $a \ngeqslant f, b \ngeqslant f, a \vee b \geqslant f, a \wedge b \ngeqslant f$ and the relation is false.

qed

Let's denote by $(O|G)_g^{g'} = O_g^{g'}$ the restriction of the order relation to G. The strict order implies that $\exists n.(O|G - Id)^n = 0$. Let $A = Id - O|G$ and $S^n = \sum_{i \in n} A^i$. We see easily that $S^n \cdot O|G = Id$. $O|G$ is therefore invertible. The matrix O_a^b is then of rank $|G|$ and all affine measures are obtained by $\mu^a = O_g^a \pi^g$ (summation restricted to G). Suppose π^g is a classical probability on G. Since the elements of O and π are positive or zero, the growth of μ is guaranteed. For $a = \perp$, the sum is empty, i.e. zero, and if $a = \top$ the sum is total and equals 1.

The result is

Theorem

Any probability is of the form

$$\mu^a = O_g^a \pi^g = \sum_{g \leqslant a} \pi^g$$

where π is a classical probability on the set G, i.e. $\pi^g \geqslant 0, \sum_g \pi^g = 1$.

[2] We'll also use index notation with Einstein's summation convention. Here, we would have $\forall t. \Delta_a^t \mu^a = 0$.

3 Antichains

It is also possible to express these probabilities using the antichains of the algebra.

Proposition

$$\mu(a) = \sum_{g \in Max(G \cap \downarrow a)} \mu(g)$$

Demonstration

$Max(G\cap \downarrow a)$ is the set of maximal elements of $\{g \leqslant a\}$. This is an antichain, since none of these maxima is comparable to any other.

If $a = \perp$, the antichain is empty, and we have $\mu(\perp) = 0$. If $a \in G$, we get the definition of μ on prime filter infs. For other elements, the Max ensures that π^g are not counted more than once.

qed

Notes

The probability μ is first defined on G, which is generally much smaller than $\mathfrak{H}$. It is extended to the other elements by the sum of its values on the antichains of G.

Let's denote the antichains of G by $AC(G)$. An order relation is defined by

$$\alpha, \beta \in AC(G) \quad \alpha \leqslant \beta \Longleftrightarrow Max(\alpha \cup \beta) = \beta$$

This order makes it a distributive lattice isomorphic to $\mathfrak{H}^{op}$, via

$$i(\alpha) = \bigvee \alpha \quad i^{-1}(a) = Max(G \cap [\perp, a])$$

4 Projectors

Let $\mathcal{H}$ be a Hilbert space of dimension $|G|$ and $\mathcal{JH}$ be the set of its orthogonal projectors. If $J, K \in \mathcal{JH}$, let $J \vee K$ be the projector on the subspace generated by the union of their images, and $J \wedge K$ that on their intersection. We also introduce a natural order relation $J \leqslant K$ given by $ImJ \subseteq ImK$.

We know that if $[J, K] = 0$, then $J \vee K = J + K - JK$ and $J \wedge K = JK$.

Let $|\beta_g\rangle$ be an orthonormal basis of $\mathcal{H}$ and $|\beta_g\rangle\langle\beta_g|$ be the associated projectors. The latter commutate and verify exactly (i).

Hence the

Theorem

Let $J : \mathfrak{H} \to \mathcal{JH}$ be given by

$$J(a) = \sum_{g \leqslant a} |\beta_g\rangle\langle\beta_g|$$

J is a monomorphism of distributive lattices.

Any probability on $\mathfrak{H}$ can be written as

$$\mu^a = \langle\psi|J(a)|\psi\rangle$$

where ψ is a vector of norm 1, in other words a *amplitude of probability* in the quantum sense.

Notes

If we denote by ψ^g the coordinates of ψ in the β_g basis, we obtain $\pi^g = |\psi^g|^2$.

Generally speaking, we don't have $J(\neg a) = Id - J(a)$. Better still, if $Id - J(a) = j(b)$, we can easily show that $a = \neg b$, $b = \neg a$, $a \vee \neg a = b \vee \neg b = \top$, $a = \neg a$ and $b = \neg b$.

5 Conditioning vs quantum reduction

The notion of *conditional probability* can be traced back to the classical definition:

$$\mu(a|b) = \frac{\mu(a \wedge b)}{\mu(b)}$$

Theorem

Let $j_b : \psi \mapsto \frac{J(b)|\psi\rangle}{\|J(b)|\psi\rangle\|}$, then $\mu(a|b) = \langle j_b\psi|J(a)|j_b\psi\rangle$

The demonstration is obvious, given the commutativity of the $J(a)$ projectors.

j_b is a *quantum reduction* operator. It is unitary, but not linear.

6 Integration and differentiation

Since π^g is a classical probability on G, we can introduce random variables $X : G \to \mathbb{R}$ and calculate integrals as $\int X d\mu = X_g \pi^g$. But $O_g^{g'}$ is invertible, and we get $\int X d\mu = X_g (\overset{-1}{O}|G)^g_{g'} \mu^{g'}$. This expression uses only the values of μ on G. $(O|G)^{-1}$ appears as a *differentiation operator*. This confirms that μ is in fact a cumulative probability.

7 Discussion

7.1 Commutative windows

The set of projectors $J(a)_{a\in\mathfrak{H}}$ with connectors $\vee, \wedge$ is a commutative subalgebra of $\mathcal{JH}$. It is a classical window on a quantum space in the sense of *Isham-Doering* [1], a context in the sense of *Kochen-Specker* [2]. The above theorems open a door to quantum modeling based on intuitionistic logic.

7.2 Square roots

The introduction of the probability amplitude as a square root is closely related to the search for non-informative *a priori* in Bayesian statistical decision theory. For example, if the aim of a sequence of experiments is to estimate a probability π^g by realization frequencies $\frac{N^g}{N}$, we can define, thanks to the *Fisher* information, an *a priori* of *Jeffreys* which has interesting invariance properties [3]. This *a priori* has a density on the simplex of dimension $|G|-1$. The vector $(\sqrt{\pi^g})_{g\in G}$ has quadratic norm 1, and the density image by $\sqrt{}$ is the *uniform* measure [4] on the sphere $\mathbb{S}^{|G|-1}$.

7.3 Non commutative extension

The β basis can be replaced by a partition of the identity $\sum_g B_g = Id$, where the B_g are orthogonal projectors of a Hilbert of sufficient dimension. Such a partition can be, for example, that of the eigenprojectors of a Hermitian operator whose eigenvalues are arbitrary, provided they are distinct for distinct projectors. This allows the coexistence of several operators that may only commute on certain subspaces, i.e. the coexistence of algebras for which the inter-algebra conjunction ($\wedge$) is only defined under certain conditions.

7.4 Superpositions

The non-prime filters f of $\mathcal{F}$ also qualify as validating systems of propositions, but they admit disjunctions that have the following property

$$\exists a, b. a \vee b \in f \ and\ a \notin f\ and\ b \notin f$$

These disjunctions, which we might call *floating*, are related to *superpositions of quantum states*. In Kripke's semantics for intuitionistic logic, these filters are not among the admissible worlds [5].

7.5 Boole algebras

When the algebra is Boolean, we retrieve classical probability because all prime filters are ultrafilters.

7.6 Stone representation

The idea of replacing atoms by prime filters is obviously directly inspired by the famous *Stone* representation theorem.

7.7 Similarities

The linking of Φ prime filters, $J(a)$ projectors and G antichains emerges from the following considerations:

If the intersection of two prime filters is prime, one is in the other.

If two orthogonal projectors commute, either they are orthogonal, or one is less than the other.

If the union of two maximal antichains is an antichain, one is in the other.

7.8 Code

All these results have been confirmed by computer. The code is written in *C++*, and the matrix calculations are performed by the *eigen3* library. [6]

8 Example

These notions have been implemented and used to present figures 1 to 4. The Heyting algebra is obtained by the antichains of an arbitrary ordered set G, and the probability μ by a uniform random draw of the probability amplitude ψ.

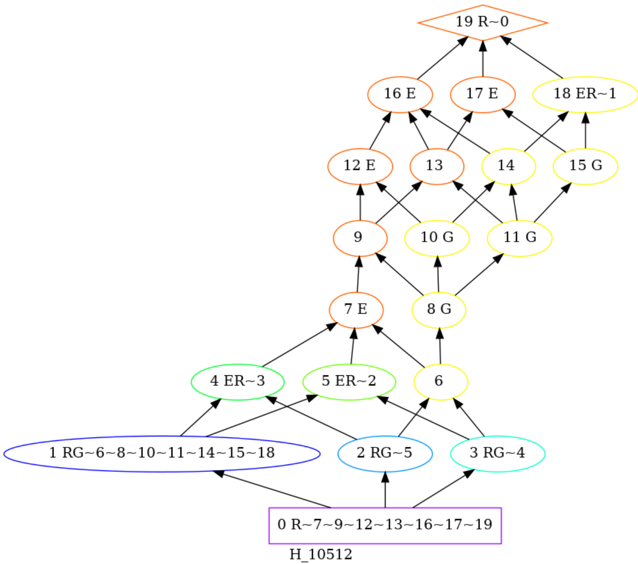

Fig. 1 Hasse diagram of a Heyting algebra.
The G are the infs of prime filters, the E the sups of prime ideals, and the R the regular elements ($a = \neg\neg a$). The $\sim$ signs describe the reciprocal image of the negation operator. Colors are iso-negation classes. In particular, the inf 7 filter is the Jacobson filter, intersection of ultrafilters, whose infs are 1, 2 and 3.

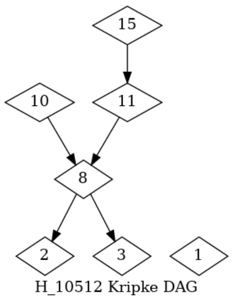

Fig. 2 Order relationship restricted to primary filter infs.
If we think of a prime filter as a world of *Kripke* [5] whose propositions are the elements, the arrows indicate the possible time courses, which go in the opposite direction to the order of the algebra. Indeed, the lower the inf, the greater the filter and the greater the number of propositions it validates. The existence of these arrows is guaranteed by the *persistence* requirement. There are two connected components, so the algebra is a product.

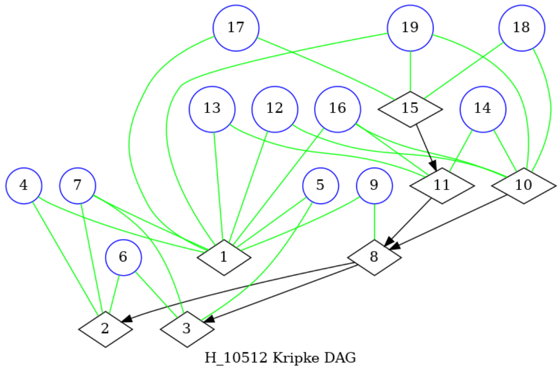

Fig. 3 Extension of μ^g by antichains.
To obtain the prob of an element outside G, we sum the probs of the ends of its outgoing arrows.

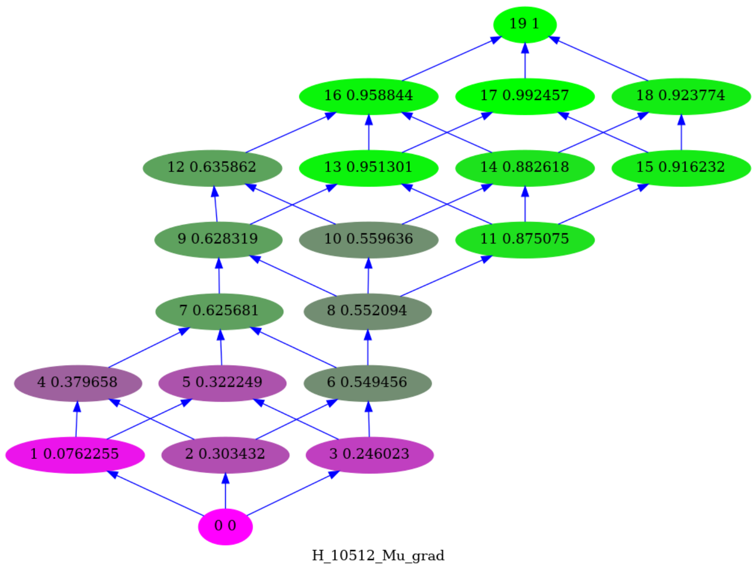

Fig. 4 Probability.
Incidentally, we can introduce a notion of information by $-\log\mu$. $\top$ contains no information because $\log\mu(\top) = \log 1 = 0$, while $-\log\mu(\bot) = -\log 0 = \infty$ suggests that infinite information would be needed to demonstrate the absurd.

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