Intuitionistic quantum observables

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0. Introduction

In a previous paper [biblio 0], we extended the notion of probability to finite Heyting algebras. Here, we propose to embed this structure in the context of quantum logic.

This embedding will allow different algebras to coexist in the same space and introduce the non-commutativity of measurement operations.

1. Probability on a Heyting algebra

Let $\mathfrak{H}$ be a finite Heyting algebra. We define its *ground* by the subset of propositions $G = \{\inf(F) \,|\, F \text{ prime filter of } \mathfrak{H}\}$. Each element $a \in \mathfrak{H}$ can be decomposed according to its *basis* $G\,.\,a = \{g \in G \,|\, g \leqslant a\}$. This gives a representation of the algebra as a topology on G whose open sets are the $G\,.\,a$ [biblio 1]. The logical connectives $\vee$, $\wedge$ are transformed into the classical operations on sets $\cup$, $\cap$. The order relation of the algebra becomes the set inclusion. The set G, equipped with this topology est isomorphic to the spectrum of the algebra and we have $a = \sup G\,.\,a$.

We place on G a classical probability $\pi(g)$, i.e. $\sum_g \pi(g) = 1, 0 \leqslant \pi(g) \leqslant 1$.

This probability extends to the entire algebra in $\mu(a) = \pi(G\,.\,a)$.

2. Standard quantum measurement

The two fondamental concepts of quantum formalism are *state* end *observable.*

The state ψ is represented by a unit vector of a certain Hilbert space $\mathscr{H}$. It evolves over time according the Schroedinger or Dirac equation [biblio 2].

The observable is represented by a Hermitian operator O. The spectral theorem gives the decomposition $O = \sum_{o \in Sp\,O} oJ_o$, where $Sp\,O$ is the spectrum of the operator, which contains the values taken by the observable. These are the eigenvalues of O. J_o is the orthogonal projector onto the eigen subspace of eigenvalue o. This set of projectors forms a *partition of the identity* $Id_{\mathscr{H}} = \sum_o J_o$. We have $o \neq o' \Longrightarrow J_o J_{o'} = 0$.

In this presentation, we associate to every observable the algebra of propositions concerning its spectrum. For standard quantum theory, this algebra is simply the power set $\mathscr{P}(Sp\,O)$. Each proposition described by a part of the spectrum $A \subseteq Sp\,O$ is matched with the projector $J_A = \sum_{o \in A} J_o$.

Each pair (J, J') of commutative projectors of the Hilbert space produces the projector $J \wedge J'$ on the intersection of their subspaces, and $J \vee J'$ on the space generated by their union. It can be easily shown that $J \wedge J' = JJ'$ and $J \vee J' = J + J' - JJ'$. These connectives are exactly the images of set operations on the spectrum: $J_{A \cup A'} = J_A \vee J_{A'}$, $J_{A \cap A'} = J_A \wedge J_{A'}$.

A measurement on a quantum system is simply a question about the spectrum of one of its observables [biblio 3]. If $A \subseteq Sp\,O$, the question A boils down to asking « Will the system give a value $o \in A$ for the observable O ? ». The answer is probabilistic. If the system is in state ψ, ils response will be *yes* with probability $p_A = \psi^\dagger J_A \psi$ and *no* with a probability $p_{\complement A} = \psi^\dagger (Id - J_A)\psi$. Furthermore, the state transforms into $\psi' = \frac{J_A \psi}{\sqrt{p_A}}$ in the *yes* case and in $\psi' = \frac{J_{\complement A}\psi}{\sqrt{1 - p_A}}$ in the *no* case.

We note that asking the same question immediately afterwards gives the same answer.

3. Intuitionistic quantum measurement

The above formalism can easily be generalised to the intuitionistic case.

An intuitionistic quantum observable O is introduced by its propositional algebra $\mathfrak{H}^O$, which is a Heyting algebra. Each element $g \in G$ of its ground is associated to a projector B_g, so that their set forms a partition of the identity of the Hilbert space $\mathscr{H}$: $Id_{\mathscr{H}} = \sum_{g \in G} B_g$.

The probability on the spectrum given a state ψ is then obtained by $\pi(g) = \psi^\dagger B_g \psi$.

The projectors B_g allow us to associate a projector with each proposition of $\mathfrak{H}^O$ by $J_a = \sum_{g \leqslant a} B_g$, and we see that $\mu(a) = \psi^\dagger J_a \psi$ is an intuitionistic probability in the sens of above.

We recover the standard case if the Heyting algebra is Boolean. Indeed, the order relation on the spectrum is then completely disconnected, and the *infs* of the prime filters are the atoms.

Intuitionistic quantum measurement provides different significations of the outcomes, because there may well be propositions called *non-exhaustive* that do not satisfy $a \vee \neg a = \top$.

For a question a, we therefore obtain :

- a *yes* answer *with probability* $\mu(a) = \psi^\dagger J_a \psi$
  and a resulting state $J_a \psi / \sqrt{\mu(a)}$
- a *no* answer with probability $\psi^\dagger (Id - J_a)\psi$
  and a resulting state $(Id - J_a \psi)/\sqrt{1 - \mu(a)}$

It is worth noting that generally, $Id - J_a \neq J_{\neg a}$. A *no* answer doesn't automatically validate $\neg a$. Moreover, asking a second time the same question gives again the same result.

The advantage of this construction is that it allows several observables to be integrated in the same Hilbert space, sharing a common state ψ. It suffîces to give each observable a partition of the identity associated with the spectrum of its algebra, for instance with help of a Hermitian operator.

The projectors of the same observable commute, but those of two different observables generally not. Nevertheless it is possible that certain projectors of two distinct partitions commute, which would give meaning to connectives $\vee$, $\wedge$ straggling two observables.

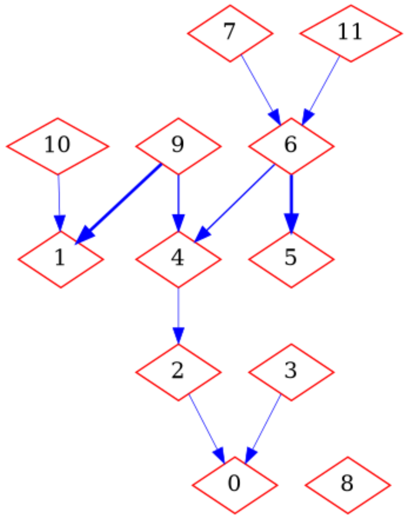
4. Example

For the algebra whose spectrum is shown beside, we have calculated the various probabilities of a *yes* answer to question a *(grey)*, a *yes* answer to question $\neg a$ *(orange)* and a residual probability $1 - \mu(a \vee \neg a)$ *(blue)*.

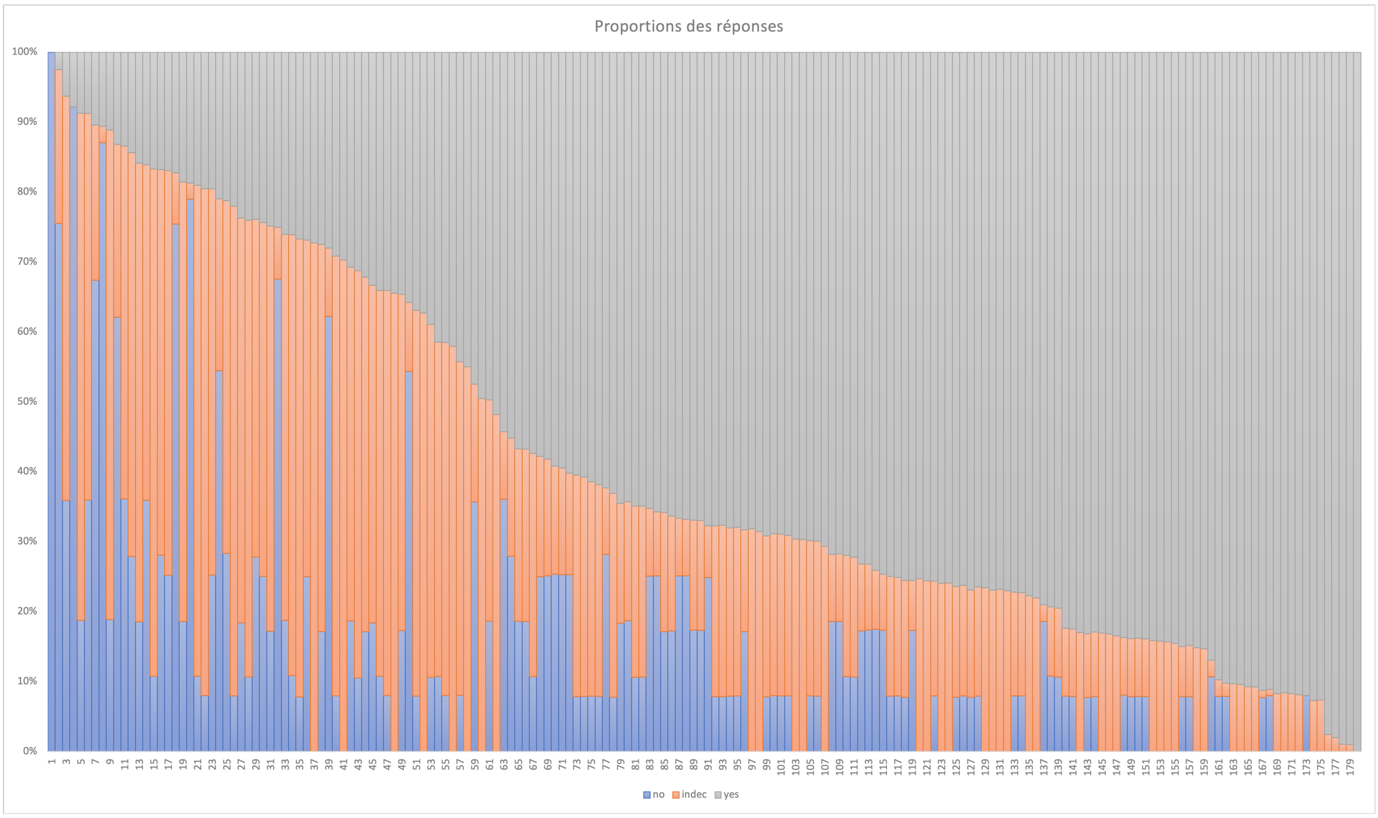
The propositions are ranked by increasing probability..

We note that the proportion of *no*'s is generally lower than that of *yes*'s. This is because negation is not surjective.

We see that the spectrum has two connected components, implying that the algebra is a product.



Obs_P_A Kripke DAG



5. Conclusion

We have generalised the notion of quantum measurement to questions whose answers may be undecidable. Each observable has been represented by a partition of the Hilbert space identity, which is a commutative window into the space of its orthogonal projectors. These windows are none other than *Kochen-Specker* contexts [biblio 4].

The embedding of Heyting algebras defining observables in Hilbert space was made possible by the convergence of two spectral concepts that are, a priori, very different: the spectra of Heyting algebras and those of Hermitian operators.

It is important to emphasise the importance of the decoupling between the sets of eigenprojectors and the associated eigenvalues. Quantum questioning is done through these sets, but does not depend on the values themselves.

Bibiography

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