Probability and Temporality (04-260125)

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Abstract

The aim of this paper is to extend the concept of probability to the framework of finite Heyting algebras. Then we will define a very interesting operator that could act as a time driver.

Keywords: probability, Heyting algebra, time driver

1 Probability on a Heyting algebra

1.1 Notations

In a finite Heyting algebra $\mathfrak{H}$, filters et ideals are always principal. We distinguish two essential subsets: G, consisting of *infs* of the prime filters, which will be called the *ground* of the algebra, and E, which contains the *prime elements*, i.e. the *sups* of the prime ideals. Since every prime filter is the complement of a prime ideal, we obtain a bijection $\gamma : E \mapsto G$ which associates the prime element with the *inf* of the corresponding prime filter: $g = \gamma(e)$.

The elements of $\mathfrak{H}$ will be called *propositions*. For each $a \in \mathfrak{H}$, we define its basis $G.a = \downarrow a \cap G$ and its co-basis $E.a = \uparrow a \cap E$. Stone's theorem [2] then states that the basis are the open sets of a topology isomorphic to $\mathfrak{H}$, and that the co-basis are the closed sets of a topology also isomorphic to $\mathfrak{H}$ [3], with the connectives defined for the inverse order. In other words:

i) $G.(a \vee b) = G.a \cup G.b \quad G.(a \wedge b) = G.a \cap G.b$

ii) $E.(a \vee b) = E.a \cap E.b \quad E.(a \wedge b) = E.a \cup E.b$

We obtain the decompositions $a = \bigvee G.a = \bigwedge E.a$

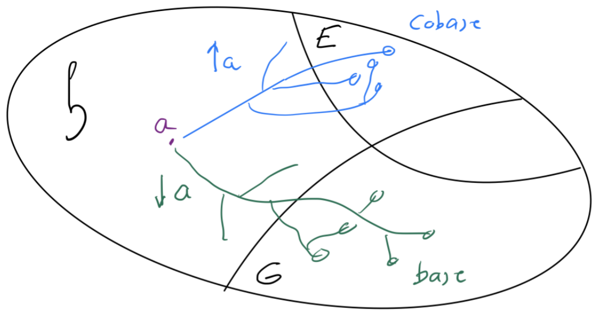

Fig. 1: Basis and co-basis

The set is also called the G-spectrum of the algebra, and is its E-spectrum.

It is easily showed that is continuous.

The order on the G-spectrum is induced by that of the algebra. This order is sufficient to characterise that of the entire algebra, since we can construct an isomorphism between the algebra and the antichains of the spectrum.

1.2 Probability construction

Classic probability is defined on a Boolean algebra $\mathfrak{B}$. We begin by defining a function $\pi : GSp\,\mathfrak{B} \to [0,1]$, such $\sum_g \pi(g) = 1$, then extend it to the entire algebra by $\pi(a) = \sum_{g \leqslant a} \pi(g)$. Since the G-spectrum is formed by the atoms of $\mathfrak{B}$, we obtain in particular $\pi(a) + \pi(\neg a) = 1$, which is consistent with the fact that $a \vee \neg a = \top$.

A probability on a Heyting algebra is defined in the same way. A classic probability is placed on the spectrum by $\pi : GSp\mathfrak{H} \to [0,1]$, then extended to the entire algebra by $\mu(a) = \sum_{g \leqslant a} \pi(g)$. The differentiation between π and μ is necessary because the order on the spectrum is no longer completely disconnected, as in the Boolean case. Indeed, μ and π only coincide on the atoms, which no longer represent the entire G-spectrum.

We find that satisfies the essential relations required for a probability:

$$\mu \geqslant 0$$
$$a \leqslant b \Longrightarrow \mu(a) \leqslant \mu(b)$$
$$\mu(\bot) = 0 \quad \mu(\top) = 1$$
$$\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$$

The definition of μ can be written in matrix form: $\mu_a = \sum_g \pi_g O_a^g$. If we restrict the indices to G , we obtain $\mu_g = \sum_{g'} \pi_{g'} O_g^{g'}$. It turns out that this matrix is invertible, which allows to easily find π from μ .

Note that $\mu(a) = \pi(G.a)$.

The structure of Heyting algebra is exactly what is needed to describe intuitionistic logic. So these probabilities will also be called *intuitionistic*.

With this definition in mind, we can identify two essential concepts. First, the concept of *conditional probability*: $\mu(a|b) = \frac{\mu(a \wedge b)}{\mu(b)}$, second that of information [6]: $Info(a) = -\log_2 \mu(a)$, which will be useful later. As is increasing, we see that the higher propositions of the algebra contain less information than the lower ones. This clearly reflects that the order relation moves toward the general. In particular, $\mu(\top) = 0$, because there is nothing to prove, and $\mu(\bot) = \infty$, which suggests that infinite information would be needed to validate the absurd.

2 Bayes revisited

Bayesian formalism allows for a reinterpretation of the spaces G et $\mathfrak{H}$.

2.1 Reminder

Bayesian statistical decision theory [5] distinguishes between two spaces: the space of the states Θ of a system, and the space X of observations on that system. During a measurement, if the state of the system is θ, its response is given by a known transition probability $p(x|\theta)$.

The observer has some knowledge of the state in the form of an a priori distribution. When he receive the response x, he can update this a priori to an a posteriori distribution using the well known formula:

$$p(d\theta|x) = \frac{p(dx|\theta)p(d\theta)}{M(dx)}$$

where $M(dx) = \int_\Theta p(dx|\theta)p(d\theta)$ is called *marginal* on the observations.

2.2 Intuitionistic transposition

In order to establish a correspondence between Bayes' formula and the definition of the intuitionistic probability, it should be noted that the variables involved in the latter are propositions, not differentials. Let us therefore reformulate the Bayesian a posteriori in terms of non-infinitesimal parts.

Let $\Gamma \subseteq \Theta$ and $A \subseteq X$, we have $p(\Gamma|A) = \frac{\int_\Gamma p(A|\theta)p(d\theta)}{M(A)}$.

This expression corresponds to the intuitionistic relation $\pi(g|G.a) = \frac{O_a^g \pi(a)}{\mu(a)}$.

This suggests making G a state space and $\mathfrak{H}$ an observation space, which leads to an a priori $\pi(g)$, an a posteriori $\pi(g|G.a)$ and a marginal $\mu(a)$. The order relation plays here the role of the system response.

It is important to highlight a key difference between the two models. The Bayesian response is an observed value, whereas the intuitionistic response is binary: O_a^g is 1 if $g \leqslant a$, 0 otherwise.

3 Irrelevance operator

We have introduced the application $\gamma : E \to G$ sopra by $\uparrow \gamma(e) = \complement \downarrow e$. The aim of this chapter is to extend it to $\mathfrak{H}$.

3.1 Relevance

Definition

$Rel_a(y) = y \to a$ is the relevance of a proposition about a proposition.

Properties

We know that $a \leqslant y \to a \leqslant \top$.

The relevance of y is therefore minimal if $y \to a = a$. In this case y is non-relevant in the usual sense to validate a. Indeed, if we use modus ponens to prove a using y, we rely on the law $y \wedge (y \to a) \leqslant a$. We must prove y and $y \to a$. If we know that $y \to a = a$, we do not need y.

If it is maximal, we have $Rel_a(y) = \top$ so $y \leqslant a$ and y automatically validates a.

The value of the relevance of about is given by

$$VR_a(y) = Info(a|y \to a) = -\log_2 \mu(a|y \to a)$$

If the relevance is minimal, its valeur is zero because $y \to a = a$. If it is maximal, its value is $Info(a)$.

It is easy to see that $VR_a(y) = Info(a) - Info(y \to a)$

3.1.1 Irrelevance filter

Let $C_0(a) = \{y|y \to a = a\}$ be the set of irrelevant propositions about a. They therefore satisfy $VR_a(y) = 0$.

Lemma

V is a filter.

Demonstration

If $y \in C_0(a)$ and $y' \geqslant y$, then $y' \to a \leqslant a$. But $a \leqslant y \to a$ give us equality. If $y, y' \in C_0(a)$, then $y \to a = a, y' \to a = a \Longrightarrow (y \wedge y') \to a = a$.

qed

Since we are in the finite case, this filter admits an infimum.

So let us set $\nu(a) = \bigwedge\{y|y \to a = a\}$, which defines the *irrelevance operator.*

It is the base axiom of irrelevant propositions for a. $\nu(a)$ is the irrelevant proposition with maximum information.

Proposition

i) $\nu(a) = \bigwedge\{y|y \to a = a\}$ is an extension of $\gamma : E \to G$

ii) $\nu(a)$ is bijective

iii) If the algebra is Boolean $\nu = \neg$

iv) $\nu_\exists \circ E. = \complement_G \circ G.$

v) $\overset{-1}{\nu} \circ G. = \complement_E \circ E.$

vi) $E.(a \to b) = Adh(\nu^{-1}G.a \cap E.b)$

vii) $G.(a \to b) = Int(\nu_\exists E.a \cap G.b)$

viii) $a \to b = \bigwedge \nu^{-1}G.a \cap E.b = \bigvee \nu_\exists E.a \cap G.b$

Lemma

$$\gamma(e) \to e = e$$

Demonstration

$\gamma(e) \to e = \bigvee\{h|\gamma(e) \wedge h \leqslant e\}$. $\gamma(e) \nleqslant e$ because $\mathfrak{H} = \uparrow \gamma(e) + \uparrow e$.

Since $\downarrow e$ is a prime ideal, $\gamma(e)$ or h must belong to it, so $h \leqslant e$ and $\bigvee\{h|\gamma(e) \wedge h \leqslant e\} \leqslant \bigvee\{h|h \leqslant e\} = e$. But then $e \leqslant \gamma(e) \to e$ gives equality.

qed

Demonstrations

i)

It must be shown that $\bigwedge\{y|y \to e = e\} = \gamma(e)$ if $e \in E$.
The lemma gives immediately $\leqslant$.
Recall that $\gamma(e) = \bigwedge \complement \downarrow e$. Let us show that $C_0(e) \subseteq \complement \downarrow e$, which will give the inverse inequality. Ab absurdo, if $a \to e = e$ and $a \leqslant e$, we have $a \to e = \top$ and $e = \top$, which is impossible because prime ideals are proper.

ii)

We have $a = \bigwedge_{e \geqslant a} g$. The e concerned are in the co-basis of a. However, it suffices to perform the conjunction on the minimal elements of the set $\{e \geqslant a\}$, which we will call its *strict co-basis.*

$$a = \bigwedge_{e' \in Min(e \geqslant a)} e'$$

The result is that we can express ν directly by the values of γ on this set.We obtain

$$\nu(a) = \bigvee_{e' \in Min(e \geqslant a)} \gamma(e')$$

Now γ is a bijection from E to G, and Min characterises a, so ν is injective. Since we are une the finite case, it is automatically bijective.

iii)

In a Boolean algebra, implication becomes $y \to a = \neg y \vee a$.

Let us translate the definition of $\nu(a) = \bigwedge\{y | y \to a = a\} : y \to a = a \iff \neg y \vee a = a \iff \neg y \leqslant a \iff y \geqslant \neg a$.

Infimum is well $\neg a$.

iv) to viii)

These relationships can be seen in the commutative diagram of Fig. 4 for the specific case of negation. This is easily extended to he case of implication.

Notes

Cl stands for closed, and *Op* for open.

The notation $\nu_\exists$ is simply the extension of ν to parts. It is the left adjoint of $\overset{-1}{\nu}$.

ν has a complex relationship with negation and complementation. ν naturally generalize the duality between prime filters and ideals to arbitrary filters and ideals. If they are prime, the *Hinterland* of Fig. 2 is empty.

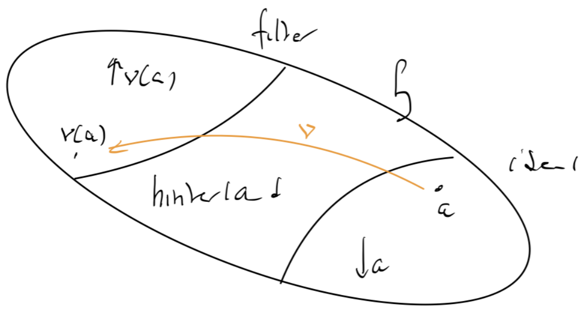

Fig. 2: Irrelevant operator

Let us define the relative relevance by $VRr_a(y) = VR_a(y)/Info(a)$. The $\uparrow \nu(a)$ gives $VRr = 0$, and the ideal $\downarrow a$ gives $VRr = 1$. The hinterland gives intermediate values. If $a \in E$, VRr only takes the values 0 or 1.

3.1.2 Geometric interpretation of implication

Let us return to viii) $a \to b = \bigwedge \nu^{-1} G.a \cap E.b = \bigvee \nu_\exists E.a \cap G.b$.

From a, we obtain its basis $G.a$ which we carry over to E by ν^{-1}. The intersection of this image with the co-basis of b gives a subset of E, which inf is $a \to b$.

The grey traces complete the adherence of the intersection. The inf is not affected by this addition.

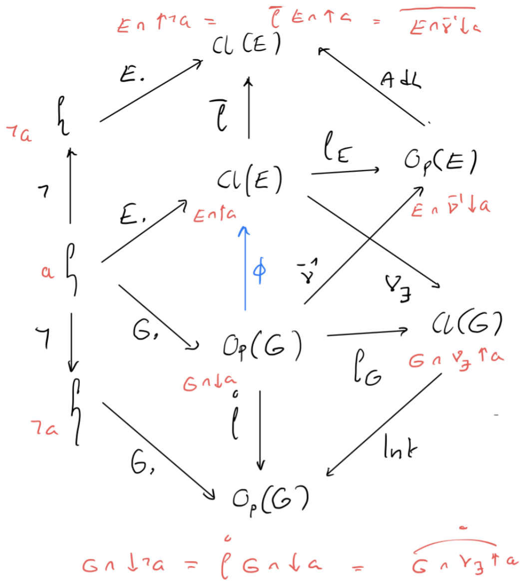

Fig. 3: Relations between topologies

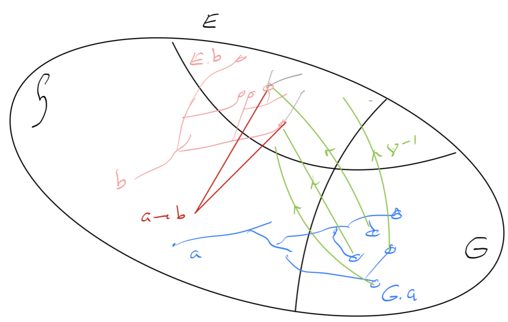

Fig. 4: Implication

3.1.3 Temporal interpretation of the irrelevance operator

If a is the ground axiom of a current reality, irrelevance makes $\nu(a)$ an ideal candidate for a later instant. Indeed, the future must be irrelevant with respect to present.

The fact that ν is bijective that there are temporal orbits in Heyting algebras.

The evolution $a \rightsquigarrow \nu(a)$ is maximal in the sense tant the next moment validates all the irrelevant propositions of the present moment. In a less extreme way, any evolution

$a \rightsquigarrow a'$ must satisfy $a' \geqslant \nu(a)$. $\uparrow \nu(a)$ can be considered as the set of potential successions of a.

Exemple

The algebra of Fig. 5 was produced by the code presented in section 4.

Note that the algebra is stratified, which is always the case. Each stratum is of a single colour. The prime elements are characterized by a single outgoing arrow in blue or red. The grounds have only one incoming arrow in green or red. There are grounds that are also prime elements.

The orbits of Fig. 6 show the iterations $\nu^t(a)$. The nodes are coloured according to their stratum.

The algebra is generated from the antichains of the spectrum. This explains the chaotic numbering of the nodes in the Hasse diagram.

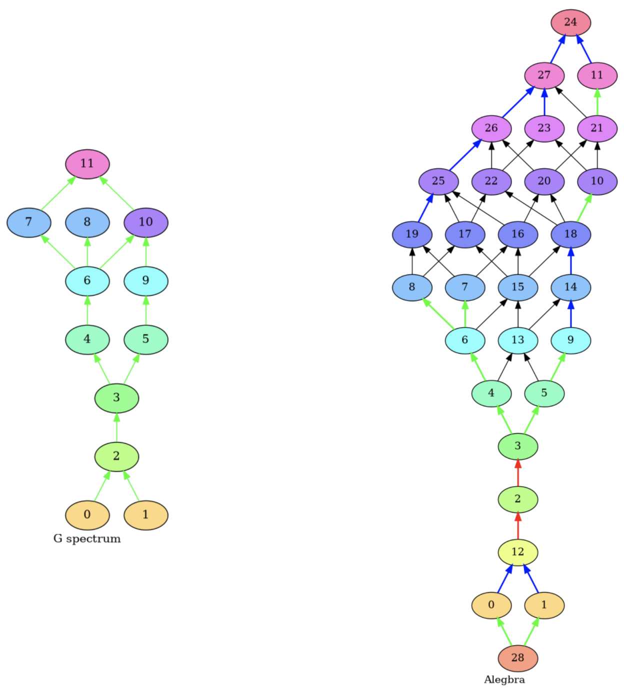

Fig. 5: Heyting algebra and its spectrum

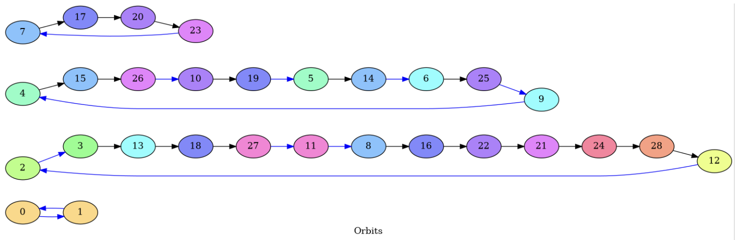

Fig. 6: ν-orbits

4 Computer implementation

All statements presented here have been tested by computer on numerous random Heyting algebras.

The construction of such an algebra is done in two steps. First, we generate an arbitrary order using a random upper diagonal matrix Tr whose coefficients are 0 or 1, and with a null diagonal. This provides a strict order generator. From this, we derive an order by $O = \min(\mathbf{1}, \sum_{k=0}^{\infty} Tr^k)$, were $\mathbf{1}$ is consisting exclusively of 1. Second, we know that the antichains of an order, equipped with the operation $\alpha \vee \beta = Max\{\alpha \cup \beta\}$ and the order relation $a \leqslant b$ given by $\alpha \vee \beta = \beta$ is a distributive lattice whose G-spectrum is the set of singletons. Since we are in the finite case, it is automatically a *Heyting* algebra.

5 Conclusion

The problem of defining a probability for a logic that admits the excluded middle runs up against the Boolean requirement of having $\pi(a \vee \neg a) = 1$. We hope to have provided a construction that circumvents this problem.

The irrelevance operator is a troubling generalisation of negation. The idea of making it a driver of time is inspired by the Greek god *Chronos*, who, in the form of *Saturn*, eats his own children.

The transition to the infinite case reveals the fundamental asymmetry of the definition of topologies, which admit arbitrary unions of open sets, but only finite intersections. Prime ideals remain principal, but this is generally not the case for filters. An extension will be the subject of another publication. It is possible to define an intuitionistic probability by passing through the prime elements, which always exist [4]. We assign a probability $\bar{\pi}(e)$ on this set and construct $\bar{\pi}(e)$. In the finite case, this detour corresponds to setting $\bar{\pi}(e) = \pi(\nu(e))$.

The evolution described by the iteration of ν can be modified by the presence of a *field* of propositions on the algebra. Iteration in a disjunctive field $a \rightsquigarrow \phi(a) \vee \nu(a)$ causes a loss of information. Such an evolution preserves the irrelevance of the future. Other possibilities include introducing a conjunctive field $a \rightsquigarrow \phi(a) \wedge \nu(a)$, which adds a hypothesis, or an implicative field $a \rightsquigarrow \phi(a) \rightarrow \nu(a)$, which subtracts one.

Paradoxically, intuitionistic logic made a comeback in the ultra-Boolean field of computer science. There are indeed programs that do not give an answer: those that do not end. A theorem by Turing tells us that there is no algorithm that can test whether a programme ends or not. The undecidable returns at the same time as the third one, which had been excluded a little too early...

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