Heyting algebras and entanglement (00-260129)

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Abstract

In this exercise, we will discuss quantum entanglement in an intuitionistic context and its evolution. This requires a definition of the tensor product, as well as the introduction of a Hamiltonian.

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1 Introduction

In a previous article [1], we defined a notion of probability for finite Heyting algebras. These algebras can be embedded in a Hilbert space in order to deal with the non-commutativity of quantum measurement.

This embedding allows us to introduce a Hamilton operator and study the temporal evolution of entanglement in a tensor algebra.

2 Ingredients

Let $\mathfrak{H}$ be a finite Heyting algebra. We denote G the inclusion of its spectrum in itself via the infima of its prime filters [2].

A probability on $\mathfrak{H}$ is first defined as a classical probability π on G, then extended to $\mathfrak{H}$ by $\mu(a) = \sum_{g \leqslant a} \pi(g)$. It is useful to introduce the order matrix $O_a^g = \chi_{[\perp,a]}(g)$ on $G \times \mathfrak{H}$. By making μ and π line vectors, we will have $\mu_a = \pi_g O_a^g$ with Einstein's summation convention. We show that the restriction of this matrix to $G \times G$ is invertible, which allows us to obtain π from μ.

An observable Ob will be an embedding of $\mathfrak{H}$ into a Hilbert space $\mathcal{H}$ in the following manner. Each element $g \in G$ is associated with an orthogonal projector B_g such that the set $(B_g)_{g\in G}$ forms a partition of the identity of $\mathcal{H}$. We then have

$$\sum_g B_g = Id, B_g B_{g'} = \delta_{gg'} B_g$$

For each element $a \in \mathfrak{H}$, we construct the orthogonal projector $J_a = B_g O^g_a$. We then note that $J_{a\vee b} = J_a + J_b - J_a J_b$, $J_{a\vee b} = J_a + J_b - J_a J_b$, $J_\perp = 0$ and $J^\top = Id$. If $\mathfrak{H}$ is not a Boolean algebra, we will generally not have $J_{\neg a} = J_a^\perp$.

The probability on $\mathfrak{H}$ will be obtained by a quantum amplitude ψ, a unit vector of $\mathcal{H}$, by calculating the matrix products:

$$\pi(g) = \psi^\dagger B_g \psi \quad \mu(a) = \psi^\dagger J_a \psi$$

An observable Ob is characterized by its *observable values* given by an injective application[1] $v : G \to \mathbb{R}$. This gives us the standard quantum model using Hermitian operators

$$\widehat{Ob} = B_g v^g$$

A projector J_a then represents a quantum measurement, which is simply a question relating to the spectrum [3]. Since $a = \bigvee_{g\leqslant a} g$, this question is expressed as *a? = Will the value of the observable appear in the set* $g \leqslant a$ *?* . If the state of the system is ψ, the Copenhagen school tells us that the answer will be *yes* with a probability $\mu = \psi^\dagger J_a \psi$, that the state will evolve towards $J_a\psi/\sqrt{\mu}$, and that the answer will be *no* with a probability $1 - \mu$ and a resulting state $J_a^\perp \psi/\sqrt{1-\mu}$.

A *yes* answer validates a. But since the algebra is only Heyting, a *no* answer doesn't necessarily validate $\neg a$. We will see below that it is nevertheless possible to validate another proposition.

Furthermore, not all questions relating to the spectrum (the set of its parts) can be asked. Since G is equipped with an order that is generally not completely disconnected, a *yes* answer to g? validates all $g' \geqslant g$. If there is an $g' > g$, there can be no answer to the question *Is the value of the observable exactly equal to g ?*

The order placed on the spectrum has, a priori, nothing to do with the order of observable values, which is that of the real numbers. It reflects a certain fuzziness on these values through the application of Birkhoff's representation theorem [4].

The advantage of this formalism is that it allows different algebras to coexist. The set of projectors induces a unitary embedding operator for each observable. In particular, the values of the observables can evolve over time by introducing a Hamiltonian on the space.

The evolution chosen here will be that of Schroedinger [5]: $\psi(t) = e^{-i\frac{\hat{H}t}{\hbar}}\psi(0)$

[1] Multiple eigenvalues are taken into account by the traces of the basic projectors

3 Tensor product

3.1 Definition and inclusions

Let $\mathfrak{H}_s, s \in S$ be a set of Heyting algebras, called *local*. The **tensor product** $\boldsymbol{\mathfrak{H}} = \otimes_s \mathfrak{H}_s$ is the algebra generated by the product spectrum:

$\boldsymbol{G} = \prod_s G_s$ equipped with the order $(g_s)_s \leqslant (g'_s)_s \iff \forall s. g_s \leqslant g'_s$.

This algebra, called *global*, is canonically constituted by the antichains of $\boldsymbol{G}$.

The projectors associated with the tensor algebra will be

$$\boldsymbol{J}_a = \boldsymbol{B}_g \ \boldsymbol{O}_a^g, g \in \boldsymbol{G}, a \in \boldsymbol{\mathfrak{H}}$$

.

We have canonical inclusions defined by $i_s : \mathfrak{H}_s \hookrightarrow \boldsymbol{\mathfrak{H}}$

$$i_s(a) = \sup\{g \in \boldsymbol{\mathfrak{H}} | \forall s. g_s \leqslant a\}$$

These inclusions define subsystems $\boldsymbol{J}_a^s = \boldsymbol{J}_{i_s(a)}, a \in \mathfrak{H}_s$.

These subsystems are subalgebras of $\boldsymbol{\mathfrak{H}}$. We can therefore solve for $\boldsymbol{B}_g^s$ the system

$$\boldsymbol{J}_a^s = \boldsymbol{B}_g^s \boldsymbol{O}_{i_s(a)}^{i_s(g)}, g \in G_s, a \in \mathfrak{H}_s$$

It is worth noting that $\boldsymbol{B}_g^s \neq \boldsymbol{B}_{i_s(g)}$. The $\boldsymbol{B}_g^s$ are generally not even in the algebra of the tensor.

3.2 Pure global states

Suppose that we choose a question a_s? in each local algebra $\mathfrak{H}_s$. Let us consider the projector $\boldsymbol{J}_{prep} = \prod_s \boldsymbol{J}_{i_s(a_s)}$, which we can call $\otimes_s \boldsymbol{J}_{i_s(a_s)}$ and which also happens to be $\bigwedge_s \boldsymbol{J}_{i_s(a_s)}$.

If the global system is in the original state ψ, the state $\boldsymbol{J}_{prep}\psi / \|\boldsymbol{J}_{prep}\psi\|$ is the one we would obtain if all the answers to the local questions were positive. This resulting state, which appears with probability $\mu_{prep} = \prod_s \psi^\dagger \boldsymbol{J}_{i_s(a_s)} \psi$, will be called **pure**[2].

4 Validation of negative responses

We have seen that a positive response to a question a? validates a in the sense that the resulting state will definitely answer *yes*, since it is then in the image of J_a (provided we do not wait too long ...).

If the answer is *no*, the system then finds itself in the image of $Id - J_a$ which may well not correspond to any proposition. Nevertheless, it is possible to obtain an optimum.

[2]These states can be obtained by rejection. Since the number of subsystems is finite, the waiting times are almost surely finite.

Proposition

$\neg^{op}a$ is the minimal proposition that is validated by $J_a^{\perp}$.

Demo

In the finite case, the opposite lattice is also a Heyting algebra, so we have:

$$J_b \geqslant J_a^{\perp} \iff J_b \geqslant Id - J_a \iff J_b(Id - J_a) = Id - J_a \iff$$
$$J_b + J_a - J_{a\wedge b} = Id \iff J_{a\vee b} = Id \iff a \vee b = \top$$

But $a \vee b = \top \iff b \geqslant \neg^{op}a$.

A *no* answer to the question a? validates any proposition implied by $\neg^{op}$.

qed

Remarks

If the algebra is Boolean, we have $\neg^{op} = \neg$.

We always have $\neg^{op}a \geqslant \neg a$, because

$$\neg^{op}a \vee a = \top \implies (\neg^{op}a \vee a) \wedge \neg a = \neg a = \neg^{op}a \wedge \neg a$$

.

5 Study of a specific case of entanglement

Consider the following experiment.

Two observers, A and B, are in the presence of a two-qubit entanglement [6].
The algebra of this system is the tensor product of two Boolean algebras with four elements:

$$\mathfrak{H} = \{\perp, a, \neg a, \top\} \otimes \{\perp', b, \neg b, \top'\}$$

This gives a Boolean algebra with 16 elements fig. 1.

From an original state ψ, chosen at random, we prepare the initial state with the projector:

$$J_{prep} = J_a \otimes J_b + J_{\neg a} \otimes J_{\neg b}$$

The interesting questions that A can ask are J_a or $J_{\neg a}$, which we will denote $J_a^{\varepsilon}, \varepsilon = \top, \perp$. Similarly, B can ask J_b^{η}. Suppose that A asks the question a? for the prepared state $\psi_{prep} = J_{prep}\psi / \|J_{prep}\psi\|$. If the answer is *yes*, then J_b remains for B, and if it is *no*, then $J_{\neg b}$ remains for B.

Symmetrically, for b? asked by B. The answers are therefore initially completely correlated.

A travels through space undergoing numerous accelerations, while B remains on Earth fig. 2. The gravitational effects of general relativity will shift the clocks, but we assume that if one of the observers measures their subsystem, the quantum repercussion occurs **at the same proper time** for the other observer.

We see that the action of A influences the state of B and vice versa, immediately in the sense of equal proper times.

A therefore asks the question $a?$ at its proper time τ_A, and B asks the question $b?$ at its proper time τ_B. When they meet again (R), they communicate their answers to each other Δ'' after the last measurement.

Two cases must be distinguished: $\tau_A < \tau_B$ and $\tau_A \geqslant \tau_B$.

In the fig. 2 and formulas below, we have denoted by τ the action of the time propagator $e^{-i\frac{H\tau}{\hbar}}$.

For example, in the case $\tau_A < \tau_B$, the probability of responses for A is given by

$$Pr_A(a? = \varepsilon, b? = \eta) = \psi^{\dagger}_{prep}\tau^{\dagger}_A J^{\varepsilon}_a \Delta^{\dagger} J^{\eta}_b \Delta''^{\dagger} \Delta'' J^{\eta}_b \Delta J^{\varepsilon}_a \tau_A \psi_{prep}$$

which reduces to

$$Pr_A(a? = \varepsilon, b? = \eta) = \psi^{\dagger}_{prep}\tau^{\dagger}_A J^{\varepsilon}_a \Delta^{\dagger} J^{\eta}_b \Delta J^{\varepsilon}_a \tau_A \psi_{prep}$$

For the observer B, we obtain

$$Pr_B(a? = \varepsilon, b? = \eta) = \psi^{\dagger}_{prep}\tau^{\dagger}_A J^{\varepsilon}_a \Delta^{\dagger} J^{\eta}_b \delta^{\dagger} \Delta''^{\dagger} \Delta'' \delta J^{\eta}_b \Delta J_A \tau_A \psi_{prep}$$

which reduces to the same expression as that of A, which is reassuring.

We note that the time-shift due to general relativity and the final delay do not come into play.

The calculation cannot be simplified further because the questions do not commute with the Hamiltonian.

The graphs fig. 3 show the evolution of the conditional probability as a function of measurement times for a Hamiltonian of average energy 2 keV.

The diagrams are digitized temporally by 200 steps of 1.25, 2.5, and 5e-20 s.

5.1 Analysis

The correlation initially decreases as a function of the time difference, then becomes periodic. There is indeed a decoherence effect.

The graph is not symmetrical, but the conditional probability is continuous, including on the diagonal $\tau_A = \tau_B$.

The derivatives on this diagonal are not continuous, but are expressed by commutators.

$$\frac{dPr_A}{\delta\tau} = \langle\psi_{prep}|\tau J_A \frac{i}{\hbar}[H, J_B] J_A \tau|\psi_{prep}\rangle$$

$$\frac{dPr_B}{\delta\tau} = \langle\psi_{prep}|\tau J_B \frac{i}{\hbar}[H, J_A] J_B \tau|\psi_{prep}\rangle$$

This result is consistent with the fact that time derivatives can be expressed by commutators with the Hamiltonian.

6 Conclusion

Quantum entanglement can easily be generalized to non-Boolean observables.

The famous spooky action at a distance mentioned by *Einstein* can be interpreted as a concordance of the proper times of the two subsystems. We can see that time shifts have no influence on the probabilities and correlations of quantum responses.

Negative responses validate the negations of the opposite algebra.

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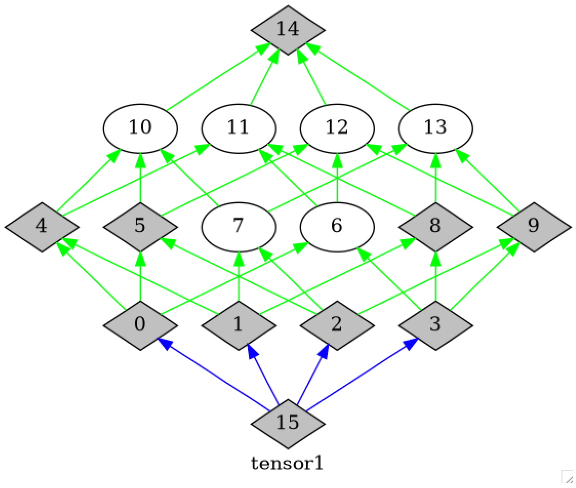

Fig. 1: Tensor algebra

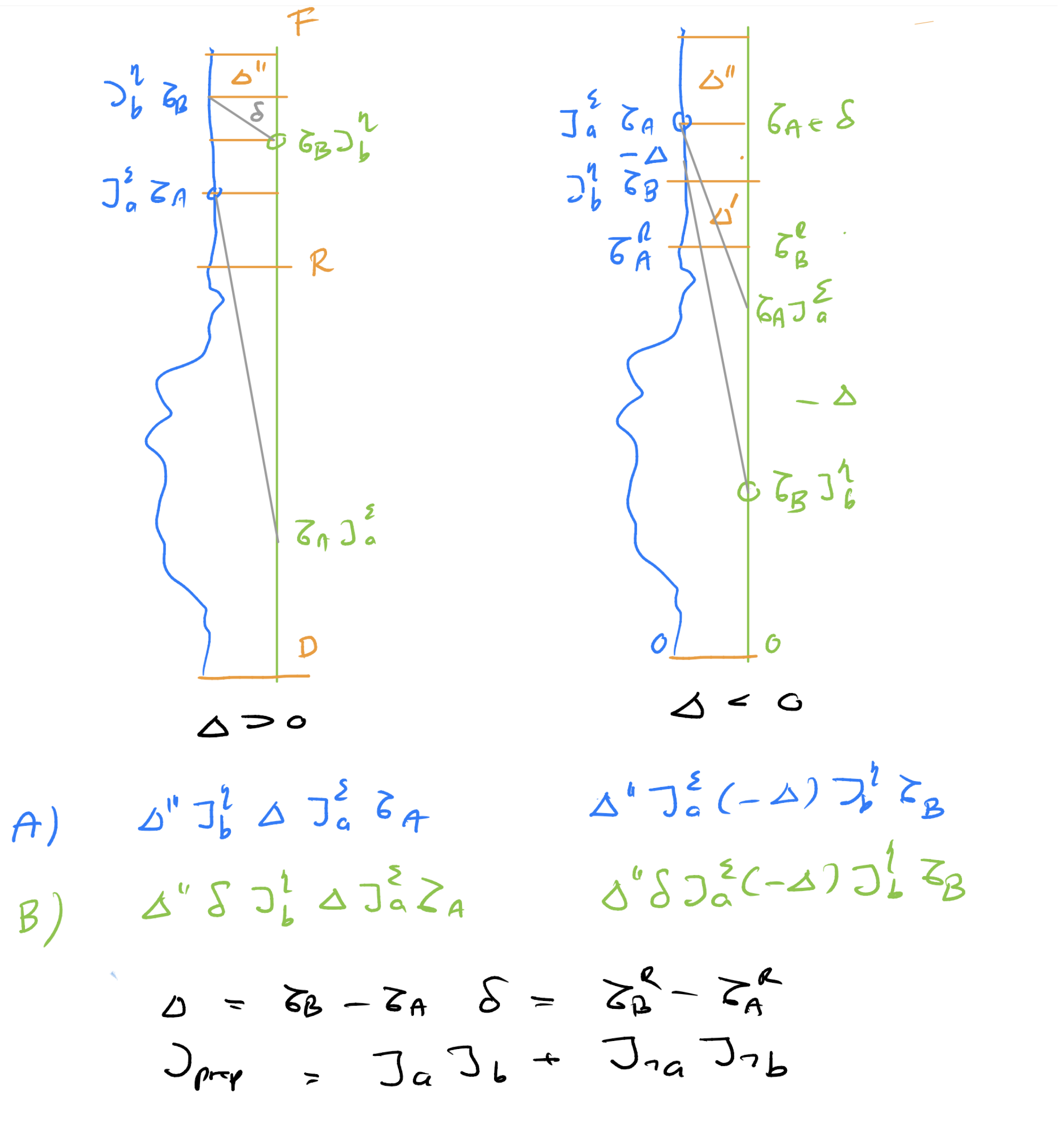

Fig. 2: Trajectories of observers A and B

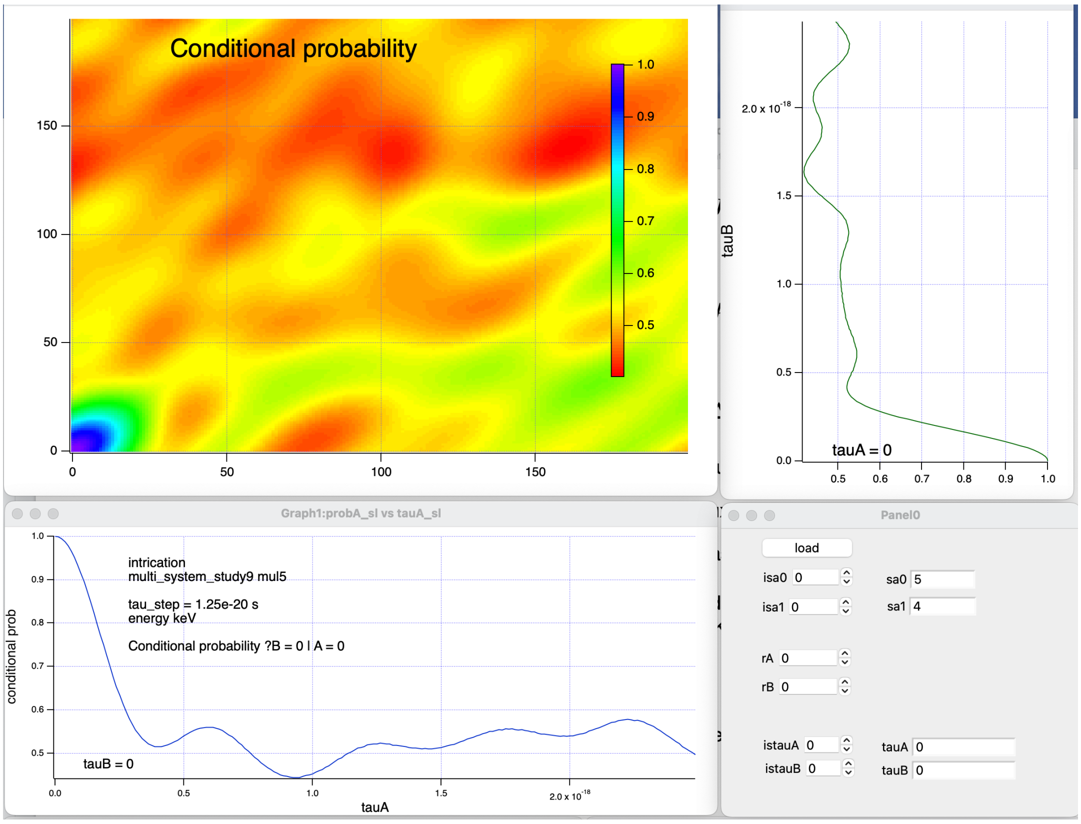

Fig. 3: Evolution of conditional intricated probabilities