Heyting algebras and entanglement

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Abstract

In this exercise, we will discuss quantum entanglement in an intuitionistic context and its evolution. This requires a definition of the tensor product, as well as the introduction of a Hamiltonian.

Keywords

Heyting algebra, intuitionistic logic, tensor product, quantum entanglement, Hilbert embedding, decoherence

Contents

1 Introduction

In a previous article [1], we defined a notion of probability for finite Heyting algebras. These algebras can be embedded in a Hilbert space in order to deal with the non-commutativity of quantum measurement.

This embedding allows us to introduce a Hamilton operator and study the temporal evolution of entanglement.

2 Brief description of the ingredients

Let $\mathfrak{H}$ be a finite Heyting algebra. We denote G the inclusion of its spectrum in itself via the infima of its prime filters [2].

A probability on $\mathfrak{H}$ is first defined as a classical probability π on G, then extended to $\mathfrak{H}$ by $\mu(a) = \sum_{g \leqslant a} \pi(g)$. It is useful to introduce the order matrix $O^g_a = \chi_{[\perp,a]}(g)$ on $G \times \mathfrak{H}$. By making μ and π line vectors, we will have $\mu_a = \pi_g O^g_a$ with Einstein's summation convention. The restriction of this matrix to $G \times G$ is invertible, which allows us to obtain π from μ.

An observable Ob will be an embedding of $\mathfrak{H}$ into a Hilbert space $\mathcal{H}$ in the following manner. Each element $g \in G$ is associated with an orthogonal projector B_g such that the set $(B_g)_{g \in G}$ forms an *orthonormal decomposition of the identity* of $\mathcal{H}$. We then have

$$\sum_g B_g = Id, B_g B_{g'} = \delta_{gg'} B_g$$

For each element $a \in \mathfrak{H}$, we construct the orthogonal projector $J_a = B_g O^g_a$. We then note that $J_{a \vee b} = J_a + J_b - J_a J_b$, $J_{a \wedge b} = J_a J_b$, $J_\perp = 0$ and $J_\top = Id$. If $\mathfrak{H}$ is not a Boolean algebra, we will generally not have $J_{\neg a} = J^\perp_a$.

The probability on $\mathfrak{H}$ will be obtained by a quantum amplitude ψ, unit vector of $\mathcal{H}$, by calculating the matrix products:

$$\pi(g) = \psi^\dagger B_g \psi \quad \mu(a) = \psi^\dagger J_a \psi$$

An observable Ob is characterized by its *observable values* given by an injective application[1] $v : G \to \mathbb{R}$. This gives us the standard quantum model using Hermitian operators

$$\widehat{Ob} = B_g v^g$$

A projector J_a then represents a quantum measurement, which is simply a question relating to the spectrum [3]. Since $a = \bigvee_{g \leqslant a} g$, this question is expressed as *a? = Will the value of the observable appear in the set $g \leqslant a$?* . If the state of the system is ψ, the Copenhagen school tells us that the answer will be *yes* with a probability $\mu = \psi^\dagger J_a \psi$, that the state will evolve towards $J_a \psi / \sqrt{\mu}$, and that the answer will be *no* with a probability $1 - \mu$ and a resulting state $J^\perp_a \psi / \sqrt{1 - \mu}$.

A *yes* answer validates a. But since the algebra is only Heyting, a *no* answer doesn't necessarily validate $\neg a$. We will see below that it is nevertheless possible to validate another proposition.

Furthermore, not all parts of the spectrum are questionable. Since G is equipped with an order that is generally not completely disconnected, a *yes* answer to g? validates all $g' \geqslant g$. If

[1] Multiple eigenvalues are taken into account by the dimensions of the images of basic projectors, which are also their traces.

there is an $g' > g$, there can be no answer to the question *Will exactly g be validated (and not g')?*

The order placed on the spectrum has, a priori, nothing to do with the order of observable values, which is that of the real numbers.

The advantage of this formalism is that it allows different algebras to coexist. The B projectors of one observable commute, but not necessarily with those of another observable. In particular, the values of the observables can evolve over time by introducing a Hamiltonian operator on the space.

The evolution chosen here will be that of Schroedinger [4]: $\psi(t) = e^{-i\frac{\hat{H}t}{\hbar}}\psi(0)$

3 Tensor product

Modelling the entanglement requires the coexistence of different observables that commute. This can be achieved as follows.

3.1 Definition and inclusions

Let $\mathfrak{H}_s, s \in S$ be a set of Heyting algebras, called *local*. The **tensor product** $\boldsymbol{\mathfrak{H}} = \otimes_s \mathfrak{H}_s$ is the algebra generated by the product spectrum:

$\boldsymbol{G} = \prod_s G_s$ equipped with the order $(g_s)_s \leqslant (g'_s)_s \iff \forall s. g_s \leqslant g'_s$.

This algebra, called *global*, is canonically constituted by the antichains of $\boldsymbol{G}$.

The projectors associated with the tensor algebra will be

$$\boldsymbol{J}_a = \boldsymbol{B}_g\ \boldsymbol{O}_a^g, g \in \boldsymbol{G}, a \in \boldsymbol{\mathfrak{H}}$$

.

We have canonical inclusions defined by $i_s : \mathfrak{H}_s \hookrightarrow \boldsymbol{\mathfrak{H}}$

$$i_s(a) = \sup\{g \in \boldsymbol{\mathfrak{H}} | g_s \leqslant a\}$$

These inclusions define subsystems $\boldsymbol{J}_a^s = \boldsymbol{J}_{i_s(a)}, a \in \mathfrak{H}_s$.

These subsystems are subalgebras of $\boldsymbol{\mathfrak{H}}$. We can therefore solve for $\boldsymbol{B}_g^s$ the system

$$\boldsymbol{J}_a^s = \boldsymbol{B}_g^s \boldsymbol{O}_{i_s(a)}^{i_s(g)}, g \in G_s, a \in \mathfrak{H}_s$$

3.2 Pure global states

Suppose that we choose a question a_s? in each local algebra $\mathfrak{H}_s$. Let us consider the projector $\boldsymbol{J}_{prep} = \prod_s \boldsymbol{J}_{i_s(a_s)}$, which we can call $\otimes_s \boldsymbol{J}_{i_s(a_s)}$ and which also happens to be $\bigwedge_s \boldsymbol{J}_{i_s(a_s)}$.

If the global system is in the original state ψ, the state $\boldsymbol{J}_{prep}\psi / \|\boldsymbol{J}_{prep}\psi\|$ is the one we would obtain if all the answers to the local questions were positive. This resulting state, which appears with probability $\mu_{prep} = \prod_s \psi^\dagger \boldsymbol{J}_{i_s(a_s)}\psi$, will be called **pure**[2].

[2] These states can be obtained by rejection. Since the number of subsystems is finite, the waiting times are almost surely finite.

4 Validation of negative responses

We have seen that a positive response to a question a? validates a in the sense that the resulting state will definitely answer *yes*, since it is then in the image of J_a (provided we do not wait too long ...).

If the answer is *no*, the system then finds itself in the image of $Id - J_a$ which may well not correspond to any proposition. Nevertheless, it is possible to obtain an optimum.

Proposition

$\neg^{op}a$ is the minimal proposition that is validated by $J_a^{\perp}$.

Demo

In the finite case, the opposite lattice is also a Heyting algebra, so we have:

$$J_b \geqslant J_a^{\perp} \iff J_b \geqslant Id - J_a \iff J_b(Id - J_a) = Id - J_a \iff$$
$$J_b + J_a - J_{a\wedge b} = Id \iff J_{a\vee b} = Id \iff a \vee b = \top$$

But $a \vee b = \top \iff b \geqslant \neg^{op}a$.

A *no* answer to the question a? validates any proposition implied by $\neg^{op}a$.

qed

Remarks

If the algebra is Boolean, we have $\neg^{op} = \neg$.

We always have $\neg^{op}a \geqslant \neg a$, because

$$\neg^{op}a \vee a = \top \implies (\neg^{op}a \vee a) \wedge \neg a = \neg a = \neg^{op}a \wedge \neg a$$

.

5 Study of some cases of entanglement

5.1 Process

Consider the following experiment.

A quantum system consists of two subsystems, A and B, whose specific propositions form (resp.) the Heyting algebras $\mathfrak{H}_A$ and $\mathfrak{H}_B$. The propositions relating to the complete system are then elements of the tensor product $\mathfrak{H}_A \otimes \mathfrak{H}_B$. Two observers, *Alice* and *Bob*, use these algebras to measure the system.

From an original quantum state ψ, chosen at random, and two propositions $a \in \mathfrak{H}_A$ and $b \in \mathfrak{H}_B$, an initial state is prepared with the projector [3]:

$$J_{prep} = J_a \otimes J_b + J_{\neg a} \otimes J_{\neg b}$$

The interesting questions that A can ask are represented by the projectors J_a or $J_{\neg a}$. Similarly, B can ask J_b ou $J_{\neg b}$. Suppose that A asks the question a? for the prepared state $\psi_{prep} = J_{prep}\psi / \| J_{prep}\psi \|$. If the answer is *yes*, then J_b remains for B, and if it is *no*, then $J_{\neg b}$ remains for B.

Symmetrically, for b? asked by B. The answers are therefore initially completely correlated.

[3] clearly inspired by Bell's first quantum state

Alice and *Bob* transport their subsystems (may be entangled photons) along distinct trajectories. At time τ_A, A measures its subsystem using the projector associated with proposition a: J_a. B does the same at its time τ_B, using J_b (see Fig. 1).

It is assumed that the effects propagate from one branch to the other **at the same proper time**.

A gravitational lens allows the two observers to meet again. They can then exchange the responses they have obtained.

Let $U_\tau = e^{-iH\tau/\hbar}$ be the Schroedinger propagator. We obtain the following probabilities for the possible responses :

If $\tau_A < \tau_B$

$$P^{<}_{ab}(\varepsilon, \eta) = \| J^{\eta}_b U_\Delta J^{\varepsilon}_a U_{\tau_A} \psi_{prep} \|^2$$

else

$$P^{>}_{ab}(\varepsilon, \eta) = \| J^{\varepsilon}_a U_{-\Delta} J^{\eta}_b U_{\tau_B} \psi_{prep} \|^2$$

Where $\varepsilon = 0, 1$, and J^ε is J if $\varepsilon = 1$ (*yes*) or $J^\perp$ if $\varepsilon = 0$ (*no*).

The final time shifts δ_A and δ_B play no role in the calculation because of the unitarity of the propagator. As H does not commute with J_a or J_b, we cannot simplify any further.

5.2 Generic simulations

5.2.1 Context

We have carried out the calculation for two cases. In the first case, the spectra of A, B and Ham are presented at Fig. 2. The tensor product grows rapidely. Here, the size of its algebra is 785.

In the second case, the spectra of observers are completely disconnected and yield Boolean algebras, see Fig. 3. The algebra becomes a hypercube in a 12 dimensions space, with 4096 nodes.

The energy of the Hamiltonian is of order of a keV.

The measuring time scale consists of 200 steps of 1e-20 s each. This results in an average action of around $\langle H \rangle d\tau \sim \hbar$

The calculations are executed by a code written in *C++* based on the *Eigen 3* matrix library.

5.2.2 Selected results

Probabilities The diagrams at Figs 4 and 5 show the probabilities that Alice's and Bob's answers will match or not (*yes* or *no* answers) when they ask questions a and b of their own algebras. The axes indicate the times at which these questions are asked.

Correlation From the contingency table of probabilities of responses, it is easy to compute the correlation between the observations of A and B. Fig. 6 shows the variation of this correlation in function of τ_A and τ_B.

Entropy With help of the consistent histories formalism [5], it is possible to compute the *von Neumann* entropy in function of the measurement times of the two observers. Let us focus on the $\tau_A < \tau_B$ case.

The initial state ψ_{prep} is replaced by the density matrix $\rho_{prep} = \psi_{prep}\psi^{\dagger}_{prep}$.

After A mesurement, this matrix becomes

$$\rho_A = J_a U_{\tau_A} \rho_{prep} U^{\dagger}_{\tau_A} J_a + J_a^{\perp} U_{\tau_A} \rho_{prep} U^{\dagger}_{\tau_A} J_a^{\perp}$$

After B measurement, this leads to

$$\rho_B = J_b U_{\Delta} \rho_A U^{\dagger}_{\Delta} J_b + J_b^{\perp} U_{\Delta} \rho_A U^{\dagger}_{\Delta} J_b^{\perp}$$

The unitary time shift δ_A doesn't change the entropy, and we get

$$Ent = Trace(-\rho_B \log_2 \rho_B)$$

The Fig. 7 show the values of entropy in function of the measurement times for the Heyting observers.

Periodicity The Hamiltonian does not depend on time, so it is not surprising to find a periodicity in the probability values when the measurement times vary. This is clearly illustrated in Fig. 8.

5.3 Qubit simulations

The standard qubit is based on Boolean logic with disconnected spectrum $\{0, 1\}$. On the other hand, the intuitionistic approach allows us to consider another type of qubit, based on the total-order spectrum $\{0 < 1\}$. This qubit will be called *Sierpinski qubit*.

The context is the same as in the previous section. The algebras and spectra are shown in Figs 9 and 10. The results are presented in Figs 11 and 12.

6 Conclusion

Quantum entanglement can easily be generalized to non-Boolean observables.

The famous spooky action at a distance mentioned by *Einstein* can be interpreted as a concordance of the proper times of the two subsystems. We can see that time shifts have no influence on the probabilities and correlations of quantum responses.

Negative responses validate the negations of the opposite algebra.

The Hamiltonian used here is of the order of a keV, which periodicities of around 1e-20 s. Bearing in mind that gravitational forces are approximately 1e-39 times weaker, these times could reach 1e19 s, that is 3e11 years for gravitational Hamiltonians.

There is still a long way to go before the relativistic metric $g_{\mu\nu}$ and the quantum operator $\hat{H}$ can be merged ...

7 Figures

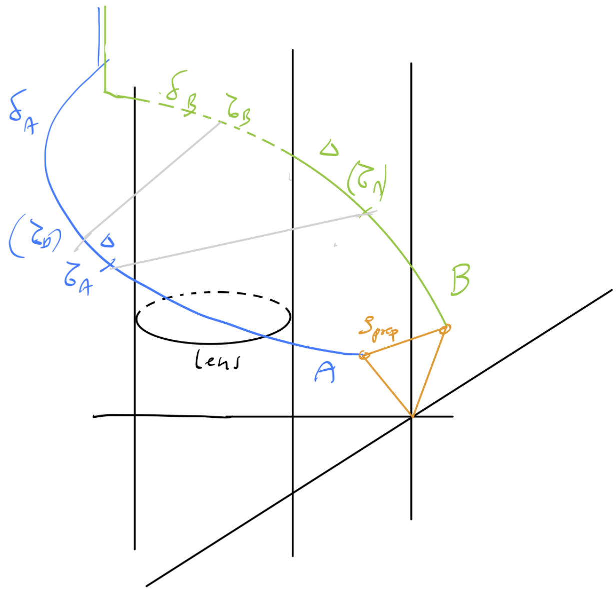

Figure 1: Trajectories of observers Alice and Bob

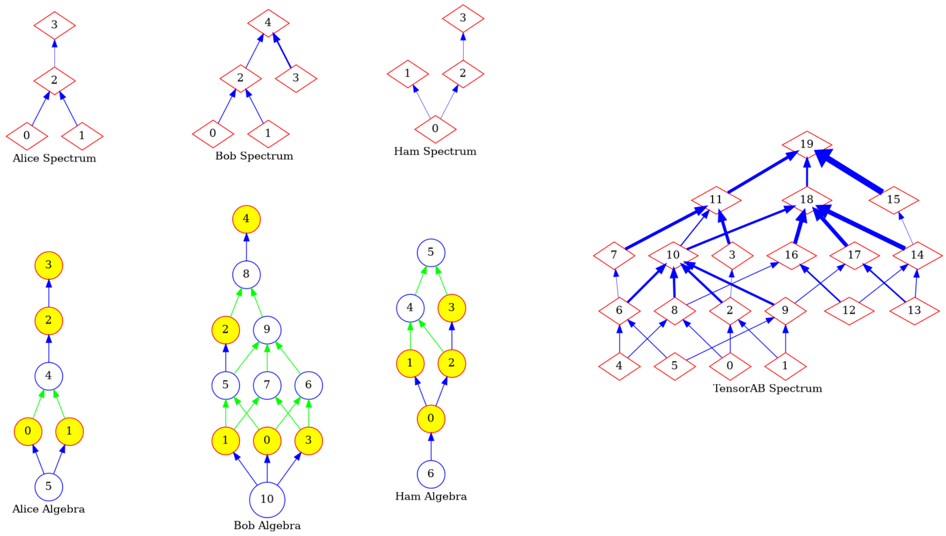

Figure 2: Spectra and algebras for Alice, Bob, Ham and tensorial for the Heyting observers

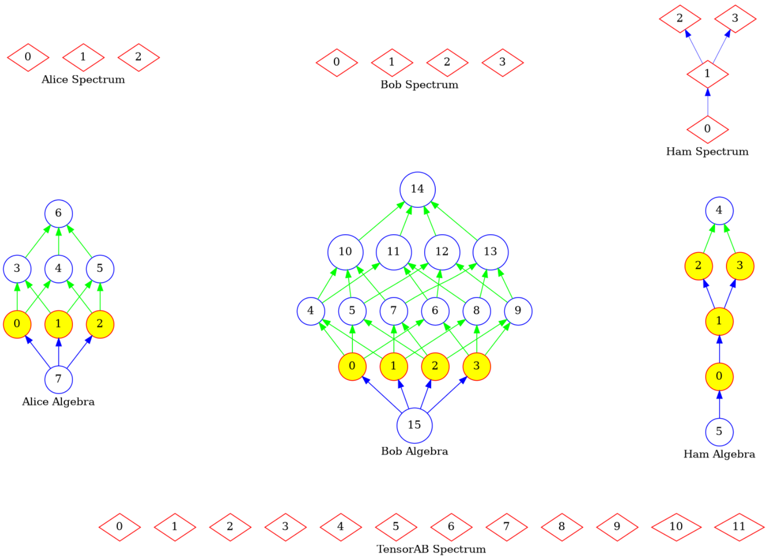

Figure 3: Spectra and algebras for Alice, Bob, Ham and tensorial the Boolean observers

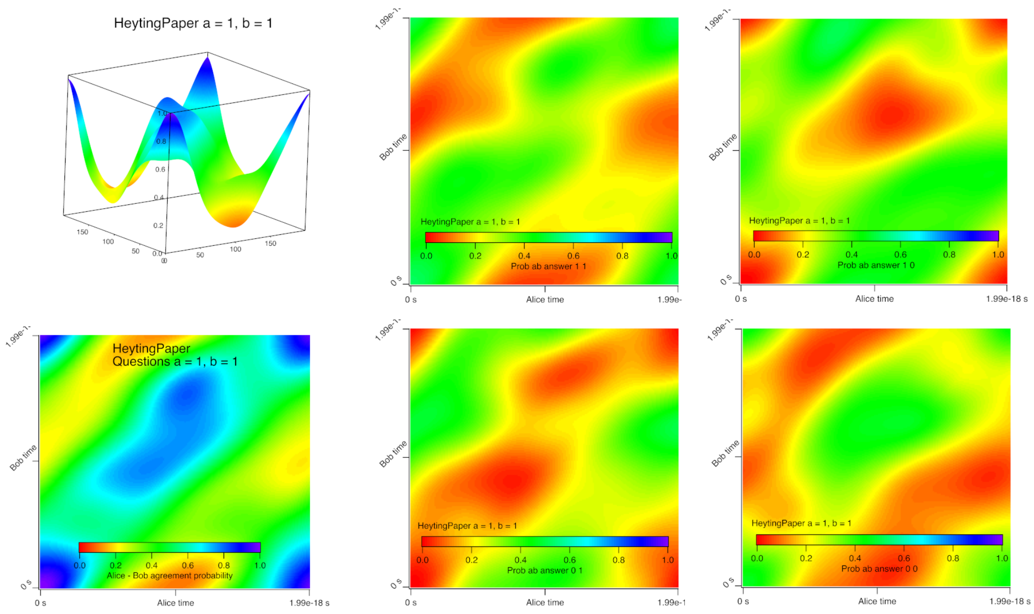

Figure 4: Results for Heyting observers

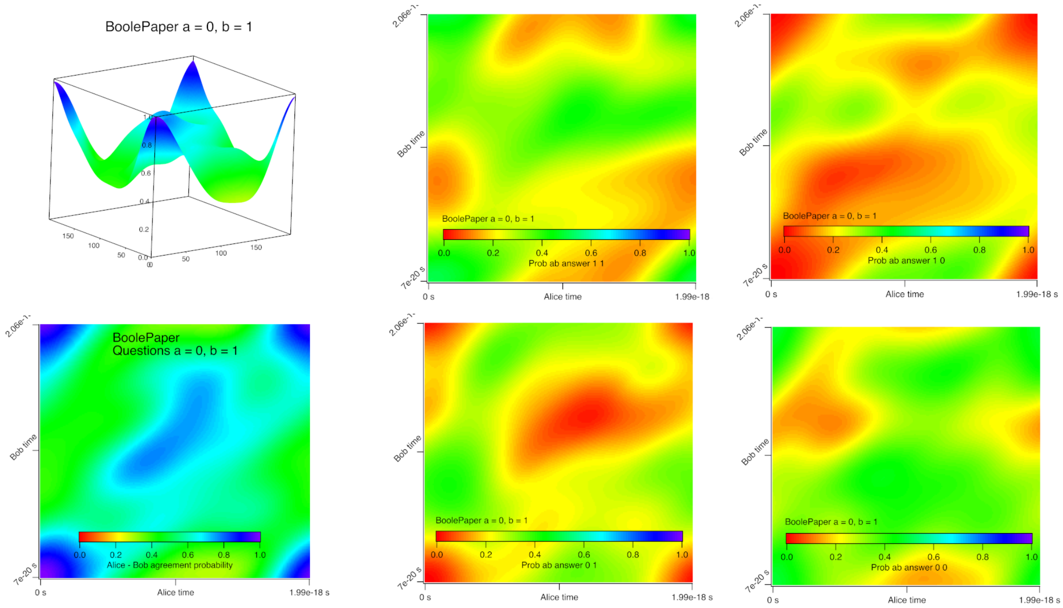

Figure 5: Results for Boole observers

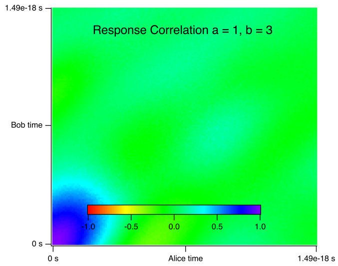

Figure 6: Correlation of the responses of A en B for Heyting observers

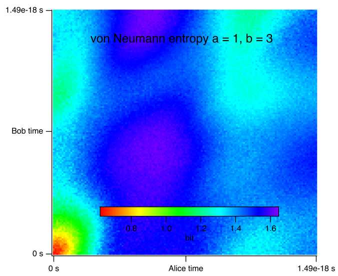

Figure 7: von Neumann entropy for Heyting observers

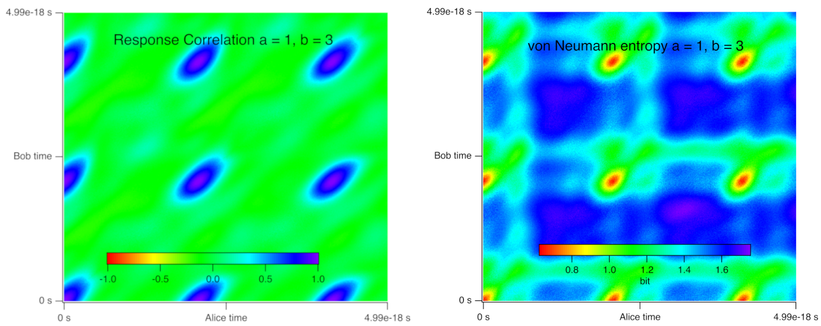

Figure 8: Periodicity for Heyting observers

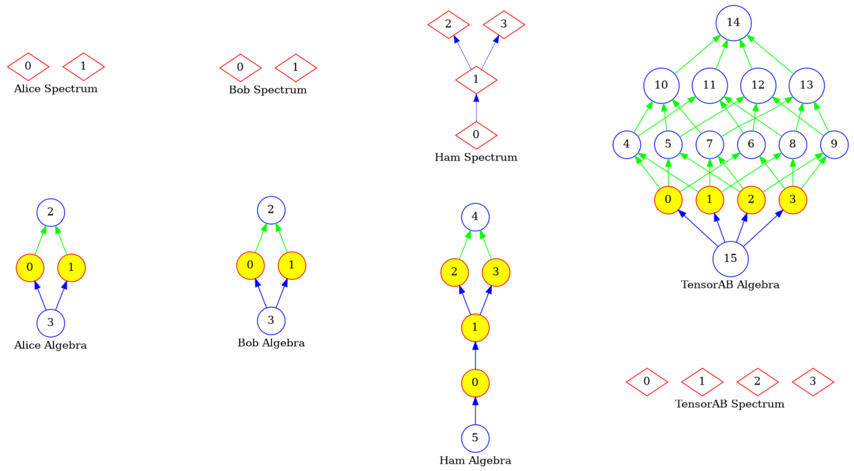

Figure 9: Spectra and algebras for the Boole qubit

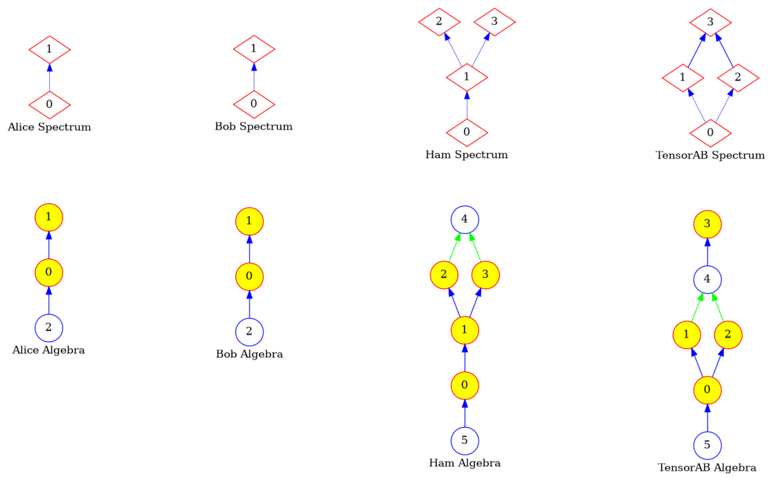

Figure 10: Spectra and algebras for the Sierpinski qubit

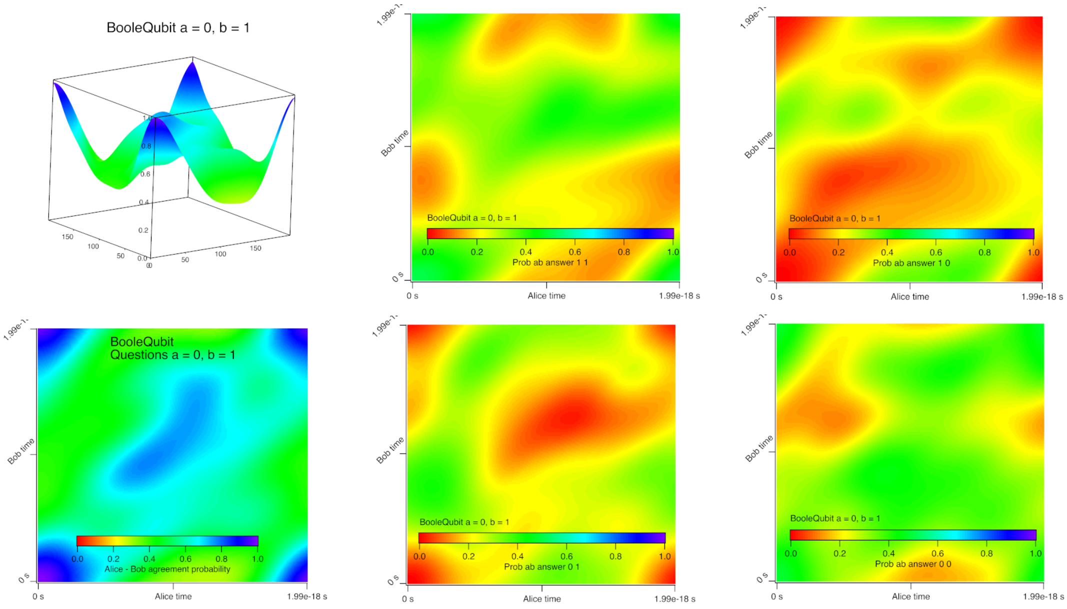

Figure 11: Results for Boole qubits

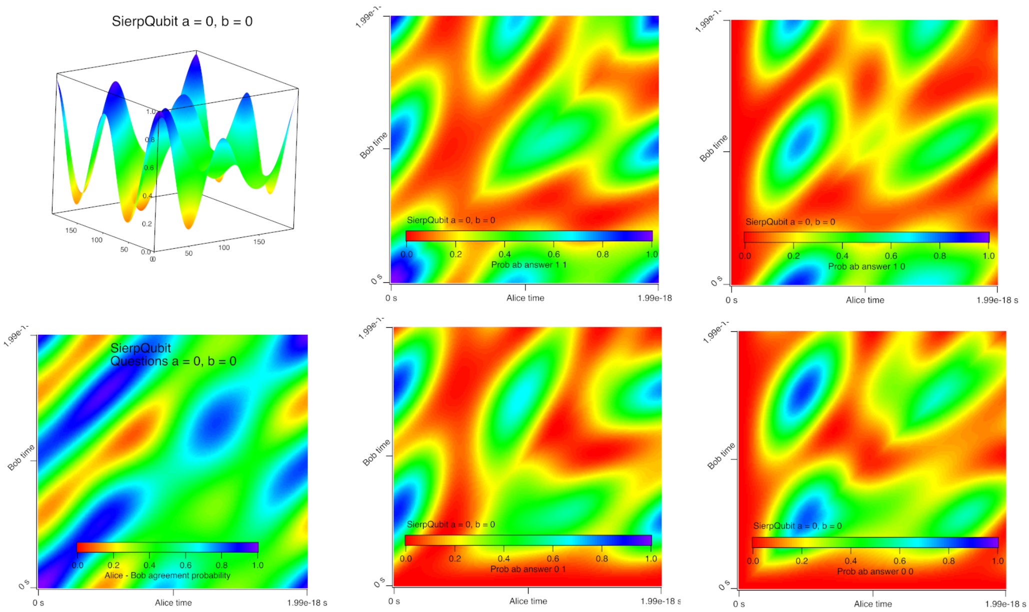

Figure 12: Results for Sierpinski qubits

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