Quantum decoherence based on Heyting algebras

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Abstract
We analys here a decoherence process in an intuitionistic logical context.

Keywords
consistent histories, decoherence, Heyting algebra, intuitionistic logic

Contents

1 Introduction

Quantum decoherence is now well understood within the framework of coherent histories [4].

Here we propose a generalisation in which the Boolean algebras of propositions associated with the observables are replaced by Heyting algebras, which we assume to be of finite cardinality.

We present an example of the evolution of such a system.

2 Definition

An observable O is defined by a finite Heyting algebra of propositions $\mathfrak{H}$, embedded in a Hilbert space $\mathcal{H}$.

Let G be the set of its *inf-prime* elements. It is well known (see [2] [3]) that every proposition can be expressed as a disjunction over this set: $a = \bigvee\{g \in G | g \leqslant a\}$.

The embedding of O into $\mathcal{H}$ is carried out as follows. To each element $g \in G$ we assign a projector B_g such that $\sum_g B_g = Id$ is an orthonormal decomposition of the identity of $\mathcal{H}$.

Let us define the *sup* and the *inf* of two projectors as $J \vee K = J + K - JK$, $J \wedge K = JK$. This results in an application $a \mapsto J_a = \bigvee\{B_g | g \in G, g \leqslant a\}$. The application J is an isomorphism of distributive lattices on its image. If we associate a real value $v(g)$ with each element g, we obtain the Hermitian operator $\hat{O} = \sum_g v(g) B_g$. It is clear that B is simply another way of expressing the spectral decomposition of $\hat{O}$.

The advantage of this formalism is that it allows different observables to be embedded in the same Hilbert space in a non-commutative manner [1]

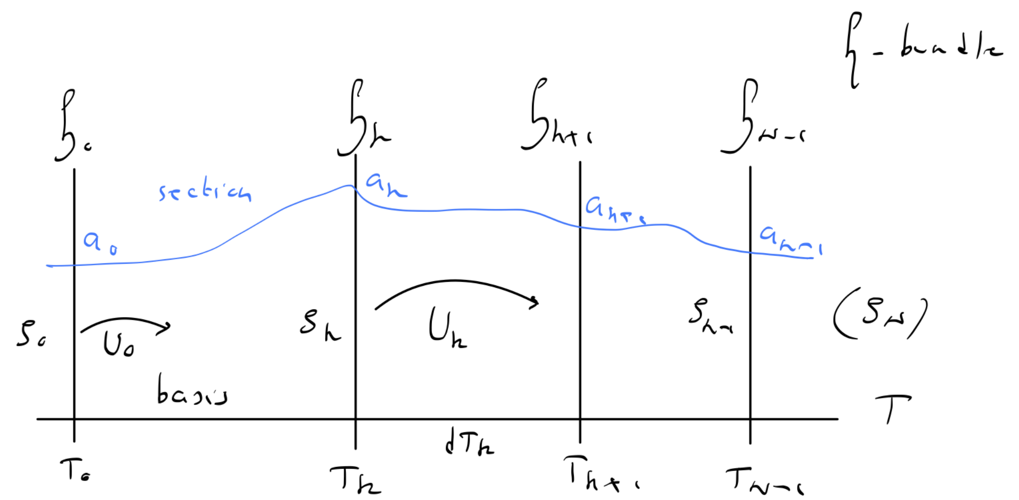

Figure 1: History of measurements

Let dT be a finite set of time intervals $\{dT_0 ... dT_{N-1}\}$. Starting from an origin T_0, we obtain the sequence of times $T_{k+1} = T_k + dT$. To each time is associated a Heyting algebra $\mathfrak{H}_k$ which describes a certain *measurement apparatus*. Given a Hamiltonian H on the space $\mathcal{H}$, it is possible to calculate the unitary evolution operators $U_k = e^{-iH\, dT_k/\hbar}$.

A history will be defined by a sequence of *measurements* (or *questions*) followed by unitary changes. Let us define:

- an initial density matrix ρ_D
- a series of *measurements* $a_k \in \mathfrak{H}_k$
- a sequential evolution $\rho_0 = \rho_D \quad \rho_{k+1} = U_k J^k_{a_k} \rho_k J^k_{a_k} U^\dagger_k$
- a final density matrix $\rho_F = \rho_N$

This is how we describe the evolution of a system ρ_k subject to perturbations a_k?.

The evolution of the sequence of decoherences is obtained in a similar way. We have:

- an initial pure density matrix $\rho_D = \psi\psi^\dagger$
- a series of *measurements* $a_k \in \mathfrak{H}_k$
- a sequential evolution $\rho_0 = \rho_D \quad \rho_{k+1} = U_k (J^k_{a_k} \rho_k J^k_{a_k} + J^{k\perp}_{a_k} \rho_k J^{k\perp}_{a_k}) U^\dagger_k$
- a final density matrix $\rho_F = \rho_N$

3 An example

3.1 Basic ingredients

This formalism has been implemented in *C++* and applied to the following situation.

The sequence of algebras is constant and constructed on the inf-spectrum:

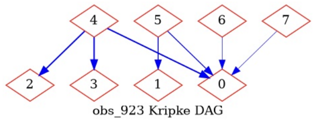

Figure 2: Measurement spectrum

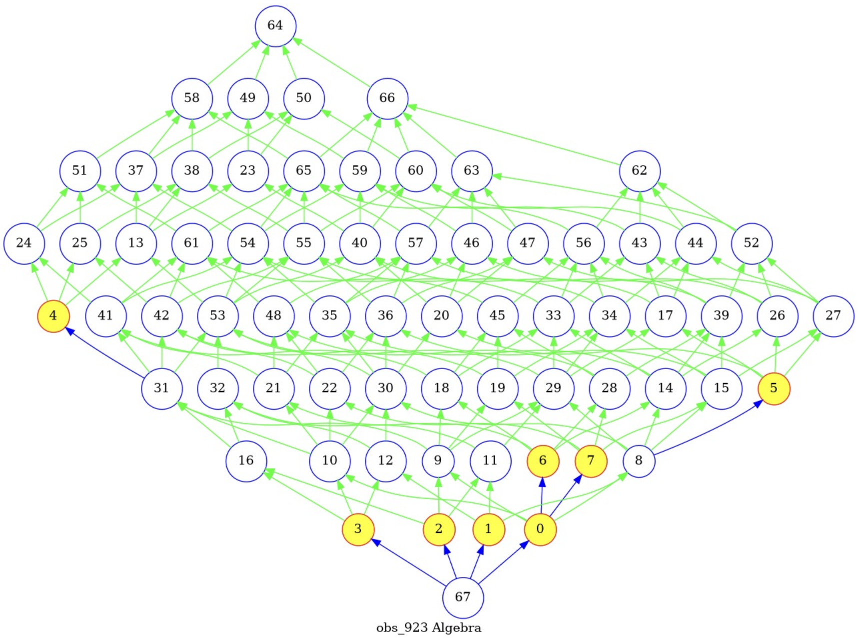

Figure 3: Measurement algebra

The Hamiltonian is as follows :

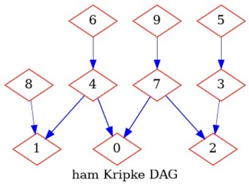

Figure 4: Hamiltonian spectrum

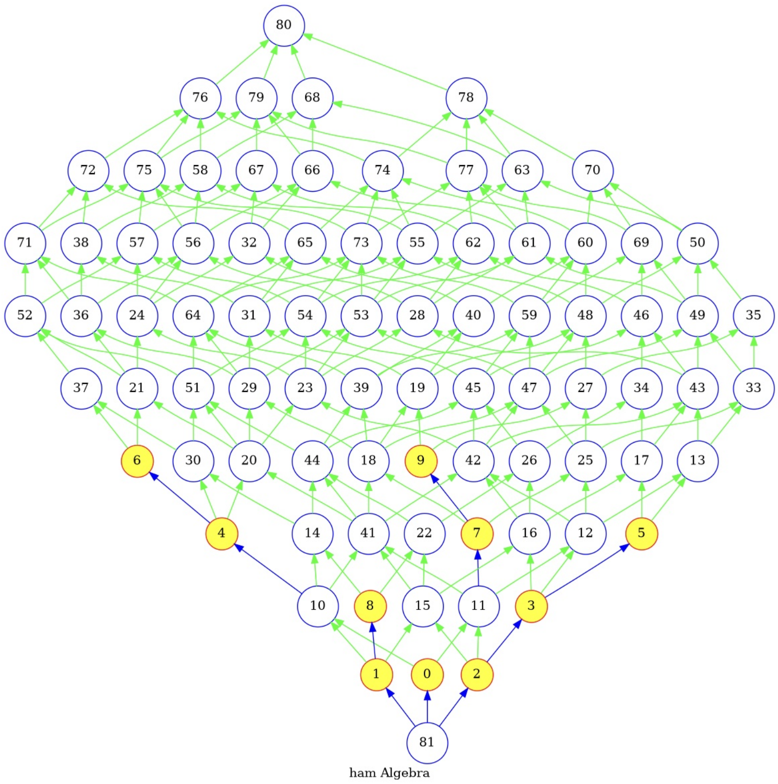

Figure 5: Hamiltonian algebra

The energy values are those of the first 10 energy levels of the hydrogen atom.

These algebras are embedded in a 120-dimensional Hilbert space.

We examined histories in which the same question is asked at each stage. We thus examined the effect of the 68 possible questions.

The time intervals are chosen uniformly at random in the interval $[0, 1]\mu s$.

3.2 Results

The graph 7 shows the evolution of von Neumann entropy for each proposition (on the x-axis) and each time step (on the y-axis, limited to 140 out of 1,000 steps). The specific curves relate to proposition 49 and the fourth time step, which is of the order of $1\mu s$.

Generally speaking, propositions that are close to $\top$ and $\bot$ yield longer decoherence times than the others. This is because the latter have more balanced projectors. Indeed, generally speaking, the result of a measurement J_a is positive with probability $\psi^\dagger J_a \psi$ and negative with probability $\psi^\dagger J_a^\perp \psi$. Entropy is maximised when these probabilities are close. It should be noted that $\top$ and $\bot$ questions have no influence on the system, which is reassuring ...

The videos *rhoJ_R_050.mp4* and *rhoJ_I_050.mp4* show the evolution of the log of the density matrix over time for question 50.

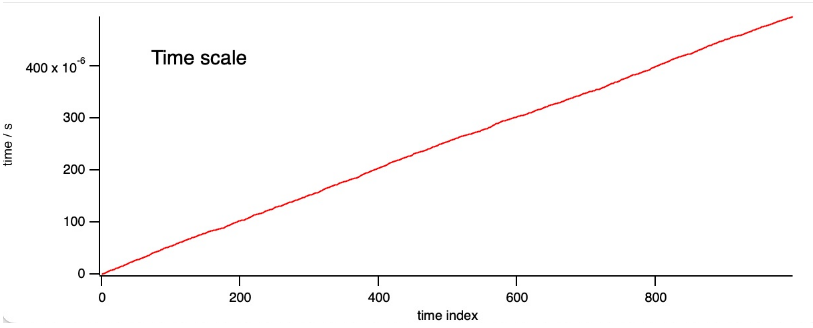

Figure 6: Time scale

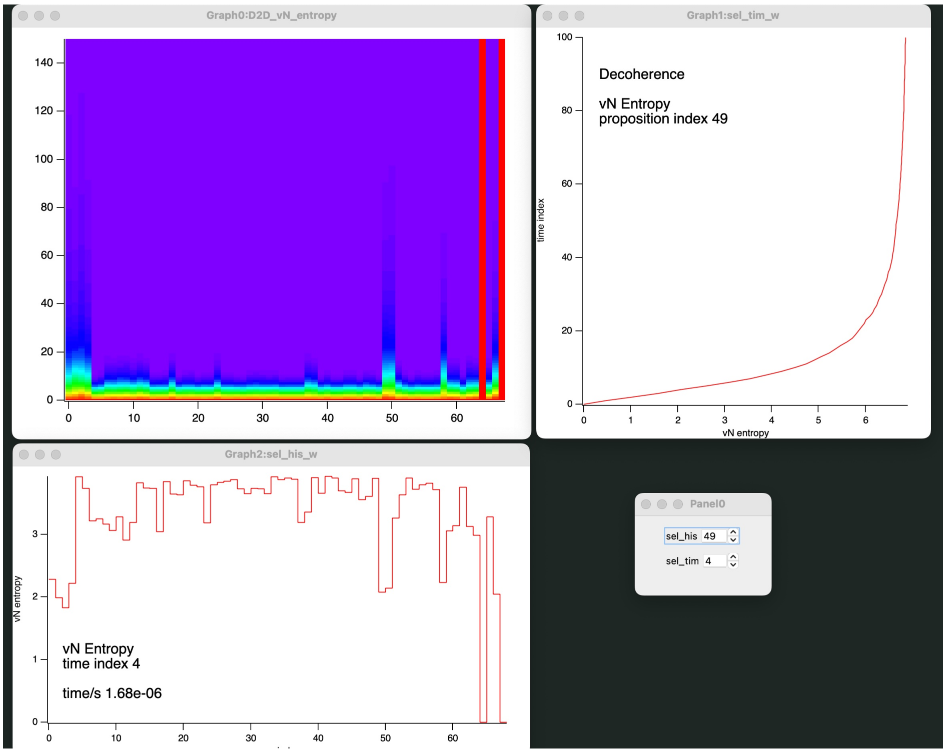

Figure 7: von Neumann entropy

The real part (which is the logarithm of the modulus) quickly becomes diagonal, whilst the imaginary part fluctuates randomly between $-\pi$ and $+\pi$, which corresponds well to a phase.

4 Conclusion

It has been shown that decoherence is an extremely rapid phenomenon.

The approach presented here defines a bundle of histories, where the base is time and the fibres are sets of measurements or states. This time is equipped with a total order. By replacing this order with a partial order, one could construct a Heyting algebra that would make time itself an observable, opening the door to relativity.

References

[1] Jean-Pascal Laedermann. Heyting algebras and entanglement, 8 2026.

[2] Dana Scott. *Continuous lattices.* Oxford University Computing Laboratory, 1971.

[3] Steven Vickers. *Topology via Logic.* Cambridge University Press, 2003.

[4] Wojciech Hubert Zurek. *Decoherence and Quantum Darwinism.* Cambridge University Press, 2025.