Quantum probability on an infinite lattice

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Abstract

An attempt to generalise intuitionistic probability to infinite lattices.

Keywords

probability, infinite lattice, intuitionistic logic

Contents

1 Introduction

In a previous paper [2], we defined a notion of probability on finite Heyting algebras. This allows us to apply this notion within a constructivist logical framework. This probability was then integrated into a Hilbert space using the concept of an observable. This integration enables several probabilistic models to coexist in a non-commutative manner.

The aim of this study is to extend this approach to infinite lattices.

Quantum theory is essentially based on separable Hilbert spaces, which may be infinite but remain countably based. Here, we shall restrict ourselves to lattices with countable spectrum.

When lattices are finite, filters and ideals are principal. This is generally not true in the infinite case. Furthermore, Kolmogorov's original definition of probability requires σ-additivity.

Here we will consider the context of continuous lattices, introduced by Dana Scott [4] in his seminal paper. Indeed, these lattices have sufficient points to allow us to base our discussion on their prime elements, and we can introduce a topology that accounts for Kolmogorov's requirement. Furthermore, these lattices are automatically Heyting algebras.

As regards the general theory, in addition to Scott's article, we drew on the works of Vickers [5], Hofmann [1] and MacLane [3].

2 Model

In a continuous lattice $\mathfrak{L}$, the prime elements are sufficient to determine the general elements. Let E be their set. For $\alpha \in \mathfrak{L}$, we can write

$$(*) \quad \alpha = \inf(E \cap \uparrow \alpha)$$

Since E is countable, we endow it with a probability $(\pi_e)_{e \in E}$ such that $\pi_e \geqslant 0$ and $\sum_e \pi_e = 1$

We extend this classical probability to $\mathfrak{L}$ as a function $\mu(\alpha) = \sum_{e \in E, e \ngeqslant \alpha} \pi_e$.

We can show that μ is increasing, verifies the modularity $\mu(\alpha) + \mu(\beta) = \mu(\alpha \wedge \beta) + \mu(\alpha \vee \beta)$ and gives the values $\mu(\top) = 1$ and $\mu(\bot) = 0$. So that is indeed what we expect from a probability.

The integration of μ into the quantum involves the concept of an **observable**. Let us consider an infinite, separable Hilbert space $\mathcal{H}$. An observable is represented by a bounded Hermitian operator $\hat{O}$. It is well known that this type of operator decomposes $\mathcal{H}$ into eigen subspaces. For an observable $\hat{O}$, we therefore construct a partition of the identity $\sum_{o \in Sp\,\hat{O}} K_o = Id_{\mathcal{H}}$ into orthogonal projectors K_o with o running through the spectrum. The degeneracy of the eigenvalues is accounted for by the dimensions of the subspaces ImK_o .

Note that these subspaces are closed and that we have $\hat{O} = \sum_{o \in Sp\,\hat{O}} v(o) K_o$, where v associate real values to the propositions of the spectrum.

The probability π to validate a prime proposition can then be calculated using a probability amplitude in the standard way by $\pi_e = \psi^\dagger K_e \psi$. For each $\alpha \in \mathfrak{L}$, we then obtain $\mu(\alpha) = \sum_{e \ngeqslant \alpha} \psi^\dagger K_e \psi$.

This allows an orthogonal projector to be associated with any proposition: $J_\alpha = \sum_{e \ngeqslant \alpha} K_e$. The set of projectors J_α is equipped with a distributive lattice structure via $J \wedge J' = JJ'$ and $J \vee J' = J + J' - JJ'$. The order relation is given by $J \leqslant J' \iff JJ' = J$, which amounts to $Im\,J \subseteq Im\,J'$. One shows that $\alpha \mapsto J_\alpha$ is an isomorphism of complete lattices.

The morphism J embeds the lattice $\mathfrak{L}$ into the Hilbert space $\mathcal{H}$. This can be done for several lattices corresponding to different observables, which do not necessarily commute.

It is nevertheless possible that some of these observables share common subspaces. This makes it possible to define partial logical connectors for non-commuting observables.

3 Sigma-additivity

The subsets $\mathcal{D}(\alpha) = \{e \in E | e \not\geq \alpha\}$ are the open sets of a topology on E that induces the *Stone* representation. $\mathcal{D}$ is an isomorphism from $\mathfrak{L}$ to the topology of E. It is easily seen that

$$\mu(\alpha) = \pi(\mathcal{D}(\alpha))$$

In other words, is nothing more than a classical probability on the Borel sets of E.

Proposition

μ is $\sigma - additive$.

Demo

Let A be a finite or countable subset of $\mathfrak{L}$, where $\alpha, \alpha' \in A, \alpha \neq \alpha' \Longrightarrow \alpha \wedge \alpha' = \perp$. We have:

$$\mu(\bigvee A) = \pi(\mathcal{D}(\bigvee A)) = \pi(\bigcup_{\alpha \in A} \mathcal{D}(\alpha)) = \sum_{\alpha \in A} \mu(\alpha)$$

because the $\mathcal{D}(\alpha)$ are disjoint.

cqfd

4 Continuity

In the classic case (i.e. Boolean), the existence of a complement allows us to express σ-additivity in a simpler way. If If M_n is a sequence of measurable subsets, we must have $(i)\, M_n \downarrow \emptyset \Longrightarrow \pi(M_n) \to 0$ or equivalently $(ii)\, M_n \uparrow E \Longrightarrow \pi(M_n) \to 1$. Let us examine what happens in the case of a lattice.

The relation $(*)$ above allows us to express the J morphism in another way. As $E \cap \uparrow \alpha = \complement\mathcal{D}(\alpha)$ we denote $Ob(\alpha) = Min(E \cap \uparrow \alpha)$ the set of its minimal elements. This is an antichain, and we can see that

$$J_\alpha = \prod_{e \in Ob(\alpha)} J_e = \prod_{e \in \complement\mathcal{D}\alpha} J_e$$

This projector is well-defined because the J_α maps onto closed spaces. We note that if $\alpha_n \uparrow \alpha$ is a chain that increases to its sup, then Ob decreases and J_{α_n} increases. The existence of the isomorphism $\mathcal{D}$ then implies that $\psi^\dagger J_{\alpha_n} \psi \to \psi^\dagger J_\alpha \psi$, since arbitrary sups are allowed. Condition (ii) is satisfied. However, condition (i) is questionable, as only finite infs are possible.

5 Validations

Ob means here *obstruction*. Since E is endowed with a classical probability π, it is possible to draw a prime element e at random according π. e will invalidate all propositions that are inferior to it. Let $h \in Hom(\mathfrak{L}, \mathbf{2})$ be such that $\downarrow e = \overset{-1}{h}(0)$. h is a validation operator which assign to every proposition in this ideal the truth value 0, thereby invalidating it. $\mu\alpha = \psi^\dagger J_\alpha \psi = \pi\mathcal{D}\alpha$ appears as the **probability of validating the proposition** (the probability of falling into $\complement\mathcal{D}\alpha$ is that of invalidating α).

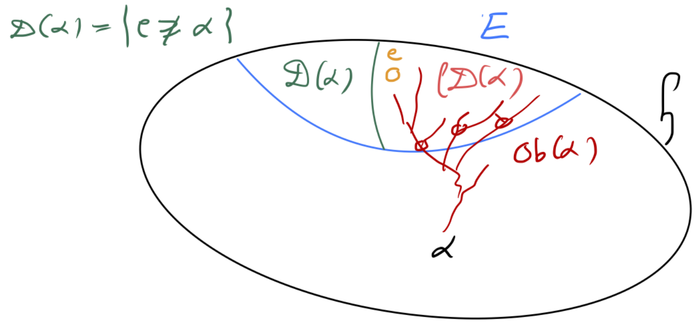

Figure 1: Obstructions and validations

It should be note that if $h(\alpha) = 1$, α is said to be **true** and if $h(\neg\alpha) = 1$, α is said to be **false**. If $h(\alpha) = 0$, α is **false** or **undecidable**. A proposition is said to be **valid** if it is usable. A false or undecidable proposition is not usable to prove anything, so it is said to be **invalid**.

6 Quantum questioning

The elementary quantum measurement consists of asking a question α? about the observable. The answer will be *yes* with probability $\psi^\dagger J_\alpha \psi$, for ψ a unit vector of $\mathcal{H}$. α is then validated and π becomes π' according to $\psi' = \frac{J_\alpha \psi}{\sqrt{\psi^\dagger J_\alpha \psi}}$. If the answer is *no*, the new state will be $\psi' = \frac{J_\alpha^\perp \psi}{\sqrt{\psi^\dagger J_\alpha^\perp \psi}}$. If the lattice is Boolean (or if the problem is regular), then $J_\alpha^\perp = J_{\neg\alpha}$, and $\neg\alpha$ is validated. Otherwise, $J_\alpha^\perp$ does not correspond to a proposition of $\mathfrak{L}$. Nevertheless, one can look for the smallest propositions in the sense $\mathfrak{L}$ of being validated by $J_\alpha^\perp$, as these are the ones that contain the most information.

Theorema

If $\neg^{op}\alpha$ exists, this is the best option if the answer is *no*. In other words:

$$\inf\{J_\beta | J_\beta \geqslant J_\alpha^\perp\} = J_{\neg^{op}\alpha}$$

Demo

$$J_\beta \geqslant Id - J_\alpha \iff J_\beta(Id - J_\alpha) = Id - J_\alpha \iff J_\alpha + J_\beta - J_\alpha J_\beta = Id \iff \alpha \vee \beta = \top \iff \beta \geqslant \neg^{op}\alpha$$

cqfd

7 Conclusion

This brief summary generalises the probability, originally defined on Boolean algebras, to constructivist σ-lattices.

The quantum embedding allows us to describe systems in which the observables do not commute.

The next step involves introducing unbounded operators. The differential equations of quantum theory yield solutions that depend on boundary conditions. These constraints lead to spectra that may be discrete or continuous. We have dealt here with the discrete case. One may ask to what extent the continuous case can be addressed by relaxing the restriction to finite infs.

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